Fuzzy Relations

Prof. Dr. Rudolf Kruse
A grey level picture interpreted as a fuzzy set
A relation among crisp sets $X_1, \ldots, X_n$ is a subset of the Cartesian Product $X_1 \times \ldots \times X_n$. It is denoted as $R(X_1, \ldots, X_n)$ or $R(X_i \mid 1 \leq i \leq n)$. So, the relation $R(X_1, \ldots, X_n) \subseteq X_1 \times \ldots \times X_n$ is set, too. The basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise}. \end{cases}$$

The membership of $(x_1, \ldots, x_n)$ in $R$ indicates whether the elements of $(x_1, \ldots, x_n)$ are related to each other or not.
The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

A **fuzzy relation** $R$ is a fuzzy set of $X_1 \times \ldots \times X_n$

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an $n$-dimensional membership array.
Let $R$ be a fuzzy relation between two sets $X = \{\text{New York City, Paris}\}$ and $Y = \{\text{Beijing, New York City, London}\}$.

$R$ shall represent relational concept “very far”.

It can be represented (subjectively) as two-dimensional membership array:

<table>
<thead>
<tr>
<th></th>
<th>NYC</th>
<th>Paris</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beijing</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>NYC</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>London</td>
<td>0.6</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Let $A_1, \ldots, A_n$ be fuzzy sets ($n \geq 2$) in $X_1, \ldots, X_n$, respectively.

The (fuzzy) *Cartesian product* of $A_1, \ldots, A_n$, denoted by $A_1 \times \ldots \times A_n$, is a fuzzy relation of the product space $X_1 \times \ldots \times X_n$.

It is defined by its membership function

$$
\mu_{A_1 \times \ldots \times A_n}(x_1, \ldots, x_n) = T(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n))
$$

for $x_i \in X_i$, $1 \leq i \leq n$.

In most applications $T = \min$ is chosen.
A special case of the Cartesian product is when $n = 2$.

Then the Cartesian product of fuzzy sets $A \in F(X)$ and $B \in F(Y)$ is a fuzzy relation $A \times B \in F(X \times Y)$ defined by

$$
\mu_{A \times B}(x, y) = T [\mu_A(x), \mu_B(y)], \text{ for all } x \in X, \text{ and } y \in Y.
$$
Example: Cartesian Product in $F(X \times Y)$ with $t$-norm = min
Projection

2 projections

Cylindrical Extension

6 projections
projection of \( \mu \)

cylindrical extension of \( \mu \)
Example

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$ and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$:

Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i,j \in \{1, 2, 3\}$.

Using this notation, all possible projections of $R$ are given below.

<table>
<thead>
<tr>
<th>$(x_1, x_2, x_3)$</th>
<th>$R(x_1, x_2, x_3)$</th>
<th>$R_{12}(x_1, x_2)$</th>
<th>$R_{13}(x_1, x_3)$</th>
<th>$R_{23}(x_2, x_3)$</th>
<th>$R_1(x_1)$</th>
<th>$R_2(x_2)$</th>
<th>$R_3(x_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0</td>
<td>0.4</td>
<td>0.9</td>
<td>1.0</td>
<td>0.5</td>
<td>1.0</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>1.0</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>0 0 2</td>
<td>0.2</td>
<td>0.9</td>
<td>0.8</td>
<td>0.2</td>
<td>1.0</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>0 1 0</td>
<td>1.0</td>
<td>1.0</td>
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<td>0.8</td>
<td>1.0</td>
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<td>1.0</td>
<td>1.0</td>
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<td>0.5</td>
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<td>0.5</td>
<td>0.5</td>
<td>0.9</td>
<td>1.0</td>
<td>0.9</td>
<td>0.9</td>
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<tr>
<td>1 0 2</td>
<td>0.1</td>
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<td>1.0</td>
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<tr>
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<td>1.0</td>
<td>0.5</td>
<td>1.0</td>
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<td>1 1 1</td>
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<td>0.5</td>
<td>0.5</td>
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<td>1.0</td>
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<td>1.0</td>
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</tr>
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</table>
Example: Detailed Calculation

Here, only consider the projection $R_{12}$:

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<tr>
<th>$(x_1, x_2, x_3)$</th>
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<tbody>
<tr>
<td>0 0 0</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>0 0 1</td>
<td>0.9</td>
<td>$(0,0)\mapsto\max[R(0, 0, 0), R(0, 0, 1), R(0, 0, 2)] = 0.9$</td>
</tr>
<tr>
<td>0 0 2</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>0 1 0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>0 1 1</td>
<td>0.0</td>
<td>$(0,1)\mapsto\max[R(0, 1, 0), R(0, 1, 1), R(0, 1, 2)] = 1.0$</td>
</tr>
<tr>
<td>0 1 2</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>1 0 0</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>1 0 1</td>
<td>0.3</td>
<td>$(1,0)\mapsto\max[R(1, 0, 0), R(1, 0, 1), R(1, 0, 2)] = 0.5$</td>
</tr>
<tr>
<td>1 0 2</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>1 1 0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>1 1 1</td>
<td>0.5</td>
<td>$(1,1)\mapsto\max[R(1, 1, 0), R(1, 1, 1), R(1, 1, 2)] = 1.0$</td>
</tr>
<tr>
<td>1 1 2</td>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>
Binary Fuzzy Relations
Consider e.g. the **membership matrix** \( R = [r_{xy}] \) with \( r_{xy} = R(x, y) \).

Its **inverse** \( R^{-1} (Y, X) \) of \( R(X, Y) \) is a relation on \( Y \times X \) defined by

\[
R^{-1}(y, x) = R(x, y) \quad \text{for all } x \in X, y \in Y.
\]

\( R^{-1} = [r_{xy}^{-1}] \) representing \( R^{-1}(y, x) \) is the transpose of \( R \) for \( R(X, Y) \)

\[
(R^{-1})^{-1} = R
\]
Consider the binary relations $P(X, Y)$, $Q(Y, Z)$ with common set $Y$.

The **standard composition** of $P$ and $Q$ is defined as

$$(x, z) \in P \circ Q \iff \exists y \in Y : (x, y) \in P \land (y, z) \in Q.$$ 

In the fuzzy case this is generalized by

$$[P \circ Q](x, z) = \sup \{\min\{P(x, y), Q(y, z)\} \mid y \in Y \}$$

for all $x \in X$ and $z \in Z$.

If $Y$ is finite, the sup operator can be replaced by max.

The standard composition is also called **max-min composition**.
Example

$$P \circ Q = R$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

\[ r_{11} = \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\} \]
\[ = \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} \]
\[ = .8 \]

\[ r_{32} = \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\} \]
\[ = \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} \]
\[ = .4 \]
The inverse of the max-min composition follows from its definition:

\[
[P(X, Y) \circ Q(Y, Z)]^{-1} = Q^{-1}(Z, Y) \circ P^{-1}(Y, X).
\]

Its associativity also comes directly from its definition:

\[
[P(X, Y)] \circ Q(Y, Z) \circ R(Z, W) = P(X, Y) \circ [Q(Y, Z) \circ R(Z, W)].
\]

Note that the standard composition is not commutative.

Matrix notation: \([r_{ij}] = [p_{ik}] \circ [q_{kj}]\) with \(r_{ij} = \max_k \min(p_{ik}, q_{kj})\).
Example: Properties of Airplanes (Speed, Height, Type)

4 possible speeds: \( s_1, s_2, s_3, s_4 \)
3 heights: \( h_1, h_2, h_3 \)
2 types: \( t_1, t_2 \)

Consider the following fuzzy relations for airplanes:

- relation \( A \) between speed and height,
- relation \( B \) between height and the type.

<table>
<thead>
<tr>
<th></th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>1</td>
<td>.2</td>
<td>0</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>.1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>.3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( t_1 )</th>
<th>( t_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( h_2 )</td>
<td>.9</td>
<td>1</td>
</tr>
<tr>
<td>( h_3 )</td>
<td>0</td>
<td>.9</td>
</tr>
</tbody>
</table>
It is also possible to define crisp or fuzzy binary relations among elements of a single set $X$.

Such a binary relation can be denoted by $R(X, X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$.

These relations are often referred to as **directed graphs** which is also an representation of them.

- Each element of $X$ is represented as a node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in $R$. 
An example of $R(X, X)$ defined on $X = \{1, 2, 3, 4\}$.

Two different representation are shown below.

\[
\begin{array}{c|cccc}
   & 1 & 2 & 3 & 4 \\
\hline
1  & .7 & 0 & .3 & 0 \\
2  & 0 & .7 & 1 & 0 \\
3  & .9 & 0 & 0 & 1 \\
4  & 0 & 0 & .8 & .5 \\
\end{array}
\]