Fuzzy Set Operators

Prof. Dr. Rudolf Kruse
Continuous Archimedian $t$-norms and $t$-conorms

Often it is possible to represent functions with several inputs by a function with only one input, e.g.

$$K(x, y) = f^{-1}(f(x) + f(y))$$

For a subclass of $t$-norms this is possible. The trick makes calculations simpler.

A $t$-norm $T$ is called

(a) **continuous** if $T$ is continuous
(b) **Archimedian** if $T$ is continuous and $T(x, x) < x$ for all $x \in ]0, 1[$.

A $t$-conorm $\perp$ is called

(a) **continuous** if $\perp$ is continuous,
(b) **Archimedian** if $\perp$ is continuous and $\perp(x, x) > x$ for all $x \in ]0, 1[$.
The concept of a pseudoinverse

**Definition**

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to $f$ is the function $f^{-1} : [c, d] \to [a, b]$ defined as

$$f^{-1}(y) = \begin{cases} 
\sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing}, \\
\sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing}.
\end{cases}$$
Definition

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\end{cases}$$
Archimedean $t$-norms

**Theorem**

A $t$-norm $\top$ is Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0, 1] \to [0, \infty)$ with $f(1) = 0$ such that

$$\top(x, y) = f^{-1}(f(x) + f(y))$$  \hspace{1cm} (1)

where

$$f^{-1}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of $f$. Moreover, this representation is unique up to a positive multiplicative constant.

$\top$ is generated by $f$ if $\top$ has representation (1).

$f$ is called additive generator of $\top$. 


Additive Generators of $t$-norms – Examples

Find an additive generator $f$ of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1, 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x, 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator $f$ of $\top_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function
Archimedean $t$-conorms

Theorem
A $t$-conorm $\perp$ is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0, 1] \to [0, \infty]$ with $g(0) = 0$ such that

$$\perp(x, y) = g^{-1}(g(x) + g(y))$$

where

$$g^{-1}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of $g$. Moreover, this representation is unique up to a positive multiplicative constant.

$\perp$ is generated by $g$ if $\perp$ has representation (2).

$g$ is called additive generator of $\perp$. 
Additive Generators of \( t \)-conorms – Two Examples

Find an additive generator \( g \) of \( \perp_{\text{Łuka}}(x, y) = \min\{x + y, 1\} \).

for instance \( g_{\text{Łuka}}(x) = x \)
then, \( g_{\text{Łuka}}^{(-1)}(x) = \min\{x, 1\} \)
thus \( \perp_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y)) \)

Find an additive generator \( g \) of \( \perp_{\text{sum}}(x, y) = x + y - x \cdot y \).

to be discussed in the exercise
hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.
Sugeno-Weber Family I

For $\lambda > -1$ and $x, y \in [0, 1]$, define

\[ T_{\lambda}(x, y) = \max \left\{ \frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0 \right\}, \]
\[ \bot_{\lambda}(x, y) = \min \{x + y + \lambda xy, 1\}. \]

$\lambda = 0$ leads to $T_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $T_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $T_{-1}$ and $\bot_{-1}$, resp.
Additive generators $f_\lambda$ of $T_\lambda$ are

$$f_\lambda(x) = \begin{cases} 
1 - x & \text{if } \lambda = 0 \\
1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.}
\end{cases}$$

$\{T_\lambda\}_{\lambda > -1}$ are increasing functions of parameter $\lambda$.

Additive generators of $\bot_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.
Fuzzy Sets Inclusion
Subset Property

For Classical Sets: $x \in A \Rightarrow x \in B,$

$A \subseteq B$

$\neg (A \subseteq B)$

For Fuzzy Sets: $x \in \mu \Rightarrow x \in \mu'$
Definition of a Fuzzy Implication

1. One way of defining $I$ is to use the property that in classical logic the propositions $a \Rightarrow b$ and $\neg a \lor b$ have the same truth values for all truth assignments to $a$ and $b$.

   If we model the disjunction and negation as $t$-conorm and fuzzy complement, resp., then for all $a, b \in [0,1]$ the following definition of a fuzzy implication seems reasonable:

   $$I(a, b) = \bot(\sim a, b).$$

2. Another way is to use the concept of a residuum in classical logic: $a \Rightarrow b$ and $\max\{x \in \{0,1\} \mid a \land x \leq b\}$ have the same truth values for all truth assignments for $a$, and $b$. If in a generalized logic the conjunction is modelled by a $t$-norm, then a reasonable generalization could be:

   $$I(a, b) = \sup\{x \in [0,1] \mid T(a, x) \leq b\}.$$
Definition of a Fuzzy Implication

3. Another proposal is to use the fact that, in classical logic, the propositions \( a \Rightarrow b \) and \( \neg a \lor (a \land b) \) have the same truth for all truth assignments.

A possible extension to many valued logics is therefore

\[
I(a, b) = \bot(\neg a, T(a, b)),
\]

where \((T, \bot, \neg)\) should be a De Morgan triplet.

So again, the classical definition of an implication is unique, whereas there is a „zoo“ of fuzzy implications.

Typical question for applications: What to use when and why?
### S-Implications

Implications based on $I(a, b) = \perp(\sim a, b)$ are called **S-implications**. Symbol $S$ is often used to denote $t$-conorms.

Four well-known $S$-implications are based on $\sim a = 1 - a$:

<table>
<thead>
<tr>
<th>Name</th>
<th>$I(a, b)$</th>
<th>$\perp(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kleene-Dienes</td>
<td>$I_{\text{max}}(a, b) = \max(1 - a, b)$</td>
<td>$\max(a, b)$</td>
</tr>
<tr>
<td>Reichenbach</td>
<td>$I_{\text{sum}}(a, b) = 1 - a + ab$</td>
<td>$a + b - ab$</td>
</tr>
<tr>
<td>Łukasiewicz</td>
<td>$I_{\text{Ł}}(a, b) = \min(1, 1 - a + b)$</td>
<td>$\min(1, a + b)$</td>
</tr>
<tr>
<td>largest</td>
<td>$I_{-1}(a, b) = \begin{cases} b, &amp; \text{if } a = 1 \ 1 - a, &amp; \text{if } b = 0 \ 1, &amp; \text{otherwise} \end{cases}$</td>
<td>$\begin{cases} b, &amp; \text{if } a = 0 \ a, &amp; \text{if } b = 0 \ 1, &amp; \text{otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>
$R$-Implications

$l(a, b) = \sup \{ x \in [0, 1] \mid T(a, x) \leq b \}$ leads to $R$-implications. Symbol $R$ represents close connection to residuated semigroup. Three well-known $R$-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to Gödel implication

  \[
  l_{\min}(a, b) = \sup \{ x \mid \min(a, x) \leq b \} = \begin{cases} 
  1, & \text{if } a \leq b \\
  b, & \text{if } a > b.
  \end{cases}
  \]

- Product leads to Goguen implication

  \[
  l_{\prod}(a, b) = \sup \{ x \mid ax \leq b \} = \begin{cases} 
  1, & \text{if } a \leq b \\
  b/a, & \text{if } a > b.
  \end{cases}
  \]

- Łukasiewicz $t$-norm leads to Łukasiewicz implication

  \[
  l_{\Ł}(a, b) = \sup \{ x \mid \max(0, a + x - 1) \leq b \} = \min(1, 1 - a + b).
  \]
**QL-Implications**

Implications based on $I(a, b) = \bot(\sim a, \top(a, b))$ are called **QL-implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**
  \[ I_Z(a, b) = \max[1 - a, \min(a, b)]. \]

- The algebraic product and sum lead to
  \[ I_p(a, b) = 1 - a + a^2 b. \]

- Using $\top_L$ and $\bot_L$ leads to **Kleene-Dienes implication** again.

- Using $\top_{-1}$ and $\bot_{-1}$ leads to
  \[ I_q(a, b) = \begin{cases} 
  b, & \text{if } a = 1 \\
  1 - a, & \text{if } a \neq 1, b \neq 1 \\
  1, & \text{if } a \neq 1, b = 1.
  \end{cases} \]
All $I$ come from generalizations of the classical implication. They collapse to the classical implication when truth values are 0 or 1. Generalizing classical properties leads to following propositions:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$ (monotonicity in 1st argument)
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$ (monotonicity in 2nd argument)
3) $I(0, a) = 1$ (dominance of falsity)
4) $I(1, b) = b$ (neutrality of truth)
5) $I(a, a) = 1$ (identity)
6) $I(a, I(b, c)) = I(b, I(a, c))$ (exchange property)
7) $I(a, b) = 1$ if and only if $a \leq b$ (boundary condition)
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ (contraposition)
9) $I$ is a continuous function (continuity)
Generator Function

A function $I : [0, 1]^2 \rightarrow [0, 1]$ satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement $\sim$ if and only if there exists a strict increasing continuous function $f : [0, 1] \rightarrow [0, \infty)$ such that $f(0) = 0$,

$$I(a, b) = f^{-1}(f(1) - f(a) + f(b))$$

for all $a, b \in [0, 1]$, and

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0, 1]$. 
Example

Consider \( f_\lambda(a) = \ln(1 + \lambda a) \) with \( a \in [0, 1] \) and \( \lambda > 0 \). Its pseudo-inverse is

\[
f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}
\]

The fuzzy \textit{negation} generated by \( f_\lambda \) for all \( a \in [0, 1] \) is

\[
n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.
\]

The resulting fuzzy implication for all \( a, b \in [0, 1] \) is thus

\[
l_\lambda(a, b) = \min \left(1, \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).
\]

If \( \lambda \in (-1, 0) \), then \( l_\lambda \) is called \textit{pseudo-Łukasiewicz implication}. 