Fuzzy Set Operators

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In set theory, **operators** are defined by **propositional logics operator**

Let $X$ be universal set (often called universe of discourse). Then we define

$$A \cap B = \{ x \in X \mid x \in A \land x \in B \}$$

$$A \cup B = \{ x \in X \mid x \in A \lor x \in B \}$$

$$A^c = \{ x \in X \mid x \notin A \} = \{ x \in X \mid \neg (x \in A) \}$$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

**Fuzzy Set Operators** can be defined by using **multivalues logics operators**
Standard Fuzzy Set Operators

\[
(\mu \land \mu')(x) := \min\{\mu(x), \mu'(x)\} \quad \text{intersection ("AND")},
\]

\[
(\mu \lor \mu')(x) := \max\{\mu(x), \mu'(x)\} \quad \text{union ("OR")},
\]

\[
\neg \mu(x) := 1 - \mu(x) \quad \text{complement ("NOT")}.
\]

\(\mu\) is subset of \(\mu'\) if and only if \(\mu \leq \mu'\).

**Theorem**

\((F(X), \land, \lor, \neg)\) is a complete distributive lattice, but no Boolean algebra.
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\mu \text{ is subset of } \mu' \text{ if and only if } \mu \leq \mu'.

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Standard Fuzzy Set Operators

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Complement (“NOT”),

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Theorem

\((F(X), \land, \lor, \neg)\) is a complete distributive lattice, but no Boolean algebra.
Fuzzy Set Complement

\[ \tilde{\mu}(x) = 1 - \mu(x) \]

\[
\begin{array}{c|ccc}
\alpha & \bar{a} = 1 - a \\
\hline
0 & 1 \\
0.3 & 0.7 \\
0.5 & 0 \\
1 & 0 \\
\end{array}
\]

\[ n : [0, 1] \rightarrow [0, 1] \]
Fuzzy Complement/Fuzzy Negation

**Definition**

Let $X$ be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \sim(\mu(x))$ where $\sim : [0, 1] \to [0, 1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

for $x, y \in [0, 1]$, $x \leq y \implies \sim x \geq \sim y$  ($\sim$ is non-increasing).

**Abbreviation:** $\sim x := \sim(x)$
Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1], x < y \implies \neg x > \neg y$ ($\neg$ is strictly decreasing)
- $\neg$ is continuous
- $\neg \neg x = x$ for all $x \in [0, 1]$ ($\neg$ is involutive)

According to conditions, two subclasses of negations are defined:

**Definition**
A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\neg x = 1 - x^2$, for instance, is strict, not strong, thus not involutive
Families of Negations

standard negation: \[ \sim x = 1 - x \]

threshold negation: \[ \sim_\theta(x) = \begin{cases} 1 & \text{if } x \leq \theta \\ 0 & \text{otherwise} \end{cases} \]

Cosine negation: \[ \sim x = \frac{1}{2} (1 + \cos(\pi x)) \]

Sugeno negation: \[ \sim_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1 \]

Yager negation: \[ \sim_\lambda(x) = (1 - x^\lambda)^{\frac{1}{\lambda}} \]

[Graphs for standard, cosine, Sugeno, and Yager negations]
Fuzzy Set Intersection and Union
Zadeh's Intersection

\( a \text{ and } b = \min(a, b) \), for all membership degrees \( a, b \)

\[(\mu_{\text{warm}} \cap \mu_{\text{hot}})(x) = \min(\mu_{\text{warm}}(x), \mu_{\text{hot}}(x)), \text{ for all real numbers } x\]
Classical Intersection and Union

Classical set intersection represents logical conjunction.
Classical set union represents logical disjunction.

Generalization from \{0, 1\} to \[0, 1\] as follows:

\[
\begin{array}{c|cc}
  x \land y & 0 & 1 \\
  \hline
  0 & 0 & 0 \\
  1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cc}
  x \lor y & 0 & 1 \\
  \hline
  0 & 0 & 1 \\
  1 & 1 & 1 \\
\end{array}
\]
Fuzzy Set Intersection and Union

Let $A, B$ be fuzzy subsets of $X$, i.e. $A, B \in F(X)$.

Their **intersection** and **union** are often defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0, 1]^2 \to [0,1]$$

$$(A \cup B)(x) = \bot(A(x), B(x)) \quad \text{where} \quad \bot : [0, 1]^2 \to [0,1].$$
Triangular Norms and Conorms

T is a *triangular norm* (t-norm) $\iff T$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm* (t-conorm) $\iff \bot$ satisfies C1-C4

**Identity Law**

T1: $T(x, 1) = x$  
C1: $\bot(x, 0) = x$

**Commutativity**

T2: $T(x, y) = T(y, x)$  
C2: $\bot(x, y) = \bot(y, x)$

**Associativity**

T3: $T(x, T(y, z)) = T(T(x, y), z)$  
C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z)$

**Monotonicity**

T4: $y \leq z$ implies $T(x, y) \leq T(x, z)$  
C4: $y \leq z$ implies $\bot(x, y) \leq \bot(x, z)$.
Triangular Norms and Conorms II

Both identity law and monotonicity respectively imply
\[ \forall x \in [0, 1] : T(0, x) = 0, \]
\[ \forall x \in [0, 1] : \bot(1, x) = 1, \]

For any t-norm \( T : T(x, y) \leq \min(x, y) \), for any t-conorm \( \bot : \bot(x, y) \geq \max(x, y) \).

\[ x = 1 \Rightarrow T(0, 1) = 0 \text{ and } \]
\[ x \leq 1 \Rightarrow T(x, 0) \leq T(1, 0) = T(0, 1) = 0 \]
De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define $t$-conorm $\perp$ by

$$\perp(x, y) = \sim \top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $\top(x, y) = \sim \perp(\sim x, \sim y), \quad x, y \in [0, 1]$. 
De Morgan Triplet II

Definition

The triplet \((\top, \bot, \sim)\) is called *De Morgan triplet* if and only if \(\top\) is \(t\)-norm, \(\bot\) is \(t\)-conorm, \(\sim\) is strong negation, \(\top, \bot\) and \(\sim\) satisfy \(\bot(x, y) = \sim \top(\sim x, \sim y)\).

In the following, some important De Morgan triplets will be shown, only the most frequently used and important ones.

In all cases, the standard negation \(\sim x = 1 - x\) is considered.
The Minimum and Maximum I

\( T_{\min}(x, y) = \min(x, y), \quad \perp_{\max}(x, y) = \max(x, y) \)

Minimum is the greatest \( t \)-norm and max is the weakest \( t \)-conorm.

\( T(x, y) \leq \min(x, y) \) and \( \perp(x, y) \geq \max(x, y) \) for any \( T \) and \( \perp \)
The Special Role of Minimum and Maximum I

$\top_{\text{min}}$ and $\bot_{\text{max}}$ play key role for intersection and union, resp. In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

**Distributivity**

$\bot_{\text{max}}(x, \top_{\text{min}}(y, z)) = \top_{\text{min}}(\bot_{\text{max}}(x, y), \bot_{\text{max}}(x, z)),$

$\top_{\text{min}}(x, \bot_{\text{max}}(y, z)) = \bot_{\text{max}}(\top_{\text{min}}(x, y), \top_{\text{min}}(x, z))$

**Continuity**

$\top_{\text{min}}$ and $\bot_{\text{max}}$ are continuous.
The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

\[ x < y \implies \top_{\min}(x, x) < \top_{\min}(y, y) \text{ and } \bot_{\max}(x, x) < \bot_{\max}(y, y). \]

Idempotency

\[ \top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x \]

Absorption

\[ \top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x \]

Non-compensation

\[ x < y < z \text{ imply } \top_{\min}(x, z) \neq \top_{\min}(y, y) \text{ and } \bot_{\max}(x, z) \neq \bot_{\max}(y, y). \]
The Minimum and Maximum II

$T_{\text{min}}$ and $\perp_{\text{max}}$ can be easily processed numerically and visually, e.g. linguistic values young and approx. 20 described by $\mu_y$, $\mu_{20}$. $T_{\text{min}}(\mu_y, \mu_{20})$ is shown below.
The Product and Probabilistic Sum

\[ T_{\text{prod}}(x, y) = x \cdot y, \quad \perp_{\text{sum}}(x, y) = x + y - x \cdot y \]
The Łukasiewicz $t$-norm and $t$-conorm

$T_{Łuka}(x, y) = \max\{0, x + y - 1\}, \quad \perp_{Łuka}(x, y) = \min\{1, x + y\}$

$T_{Łuka}, \perp_{Łuka}$ are also called **bold intersection** and **bounded sum**.
The Drastic Product and Sum

\[ T_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \bot_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases} \]

\( T_{-1} \) is the weakest \( t \)-norm, \( \bot_{-1} \) is the strongest \( t \)-conorm.

\( T_{-1} \leq T \leq T_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1} \) for any \( T \) and \( \bot \).
Examples of Fuzzy Intersections

Note that all fuzzy intersections are contained within upper left graph and lower right one.
Examples of Fuzzy Unions

$t$-conorm $\perp_{\text{max}}$

$t$-conorm $\perp_{\text{sum}}$

$t$-conorm $\perp_{\text{Łuka}}$

$t$-conorm $\perp_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.