Fuzzy Set Operators

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In set theory, **operators** are defined by **propositional logics operator**

Let $X$ be universal set (often called universe of discourse). Then we define

$$A \cap B = \{ x \in X \mid x \in A \land x \in B \}$$

$$A \cup B = \{ x \in X \mid x \in A \lor x \in B \}$$

$$A^c = \{ x \in X \mid x \notin A \} = \{ x \in X \mid \neg (x \in A) \}$$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

**Fuzzy Set Operators** can be defined by using **multivalues logics operators**
Standard Fuzzy Set Operators

\[
\begin{align*}
(\mu \land \mu')(x) & := \min\{\mu(x), \mu'(x)\} \quad \text{intersection ("AND"),} \\
(\mu \lor \mu')(x) & := \max\{\mu(x), \mu'(x)\} \quad \text{union ("OR"),} \\
\lnot \mu(x) & := 1 - \mu(x) \quad \text{complement ("NOT").}
\end{align*}
\]

\mu \text{ is subset of } \mu' \text{ if and only if } \mu \leq \mu'.

**Theorem**

\((\mathcal{F}(X), \land, \lor, \lnot) \text{ is a complete distributive lattice, but no Boolean algebra.}\)
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Fuzzy Set Complement

\[ \tilde{\mu}(x) = 1 - \mu(x) \]

\[
\begin{array}{c|cccc}
\alpha & 0 & 0.3 & 0.7 & 1 \\
\hline \\
0 & 0 & 0 & 0 & 1 \\
0.3 & 0 & 0.3 & 0.7 & 1 \\
0.7 & 0 & 0.3 & 0.7 & 1 \\
1 & 0 & 0.3 & 0.7 & 1 \\
\end{array}
\]

\[ N: [0,1] \rightarrow [0,1] \]
**Fuzzy Complement/Fuzzy Negation**

**Definition**

Let $X$ be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \bar{\sim}(\mu(x))$ where $\sim : [0, 1] \rightarrow [0, 1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

for $x, y \in [0, 1]$, $x \leq y \Rightarrow \sim x \geq \sim y$ (\(\sim\) is non-increasing).

**Abbreviation:** $\sim x := \sim(x)$
Strict and Strong Negations

Additional properties may be required

• \( x, y \in [0, 1], \ x < y \implies \sim x > \sim y \) (\( \sim \) is strictly decreasing)
• \( \sim \) is continuous
• \( \sim \sim x = x \) for all \( x \in [0, 1] \) (\( \sim \) is involutive)

According to conditions, two subclasses of negations are defined:

**Definition**

A negation is called **strict** if it is also strictly decreasing and continuous. A strict negation is said to be **strong** if it is involutive, too.

\( \sim x = 1 - x^2 \), for instance, is strict, not strong, thus not involutive
Families of Negations

standard negation: $\sim x = 1 - x$

threshold negation: $\sim_{\theta}(x) = \begin{cases} 1 & \text{if } x \leq \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2} (1 + \cos(\pi x))$

Sugeno negation: $\sim_{\lambda}(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$

Yager negation: $\sim_{\lambda}(x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$
Fuzzy Set Intersection and Union
warm and hot ?
Zadeh’ Intersection
a and b = min (a, b), for all membership degrees a, b
$(\mu_{\text{warm}} \cap \mu_{\text{hot}})(x) = \min(\mu_{\text{warm}}(x), \mu_{\text{hot}}(x))$, for all real numbers x
Classical Intersection and Union

Classical set intersection represents logical conjunction.
Classical set union represents logical disjunction.
Generalization from \( \{0, 1\} \) to \([0, 1]\) as follows:

\[
\begin{array}{c|cc}
\land & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\lor & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Fuzzy Set Intersection and Union

Let $A, B$ be fuzzy subsets of $X$, i.e. $A, B \in F(X)$. Their **intersection** and **union** are often defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x))$$

where

$$(A \cup B)(x) = \bot(A(x), B(x))$$

where

$\top : [0, 1]^2 \to [0, 1]$  
$\bot : [0, 1]^2 \to [0, 1]$. 
Triangular Norms and Conorms

$\top$ is a triangular norm (t-norm) $\iff$ $\top$ satisfies conditions T1-T4

$\bot$ is a triangular conorm (t-conorm) $\iff$ $\bot$ satisfies C1-C4

Identity Law

**T1:** $\top(x, 1) = x$  
**C1:** $\bot(x, 0) = x$

Commutativity

**T2:** $\top(x, y) = \top(y, x)$  
**C2:** $\bot(x, y) = \bot(y, x)$

Associativity

**T3:** $\top(x, \top(y, z)) = \top(\top(x, y), z)$  
**C3:** $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z)$

Monotonicity

**T4:** $y \leq z$ implies $\top(x, y) \leq \top(x, z)$  
**C4:** $y \leq z$ implies $\bot(x, y) \leq \bot(x, z)$.
Both identity law and monotonicity respectively imply
\[ \forall x \in [0, 1] : T(0, x) = 0, \]
\[ \forall x \in [0, 1] : \bot(1, x) = 1, \]

For any \( t \)-norm \( T : T(x, y) \leq \min(x, y) \), for any \( t \)-conorm \( \bot : \bot(x, y) \geq \max(x, y) \).

\[ x = 1 \Rightarrow T(0, 1) = 0 \] and
\[ x \leq 1 \Rightarrow T(x, 0) \leq T(1, 0) = T(0, 1) = 0 \]
De Morgan Triplet I

For every $T$ and strong negation $\sim$, one can define $t$-conorm $\bot$ by

$$\bot(x, y) = \sim T(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $T(x, y) = \sim \bot(\sim x, \sim y), \quad x, y \in [0, 1]$. 
De Morgan Triplet II

Definition

The triplet \((\top, \bot, \sim)\) is called \textit{De Morgan triplet} if and only if \(\top\) is \(t\)-norm, \(\bot\) is \(t\)-conorm, \(\sim\) is strong negation, \(\top, \bot\) and \(\sim\) satisfy \(\bot(x, y) = \sim \top(\sim x, \sim y)\).

In the following, some important De Morgan triplets will be shown, only the most frequently used and important ones.

In all cases, the standard negation \(\sim x = 1 - x\) is considered.
The Minimum and Maximum I

\[ T_{\min}(x, y) = \min(x, y), \quad \bot_{\max}(x, y) = \max(x, y) \]

Minimum is the greatest \( t \)-norm and max is the weakest \( t \)-conorm.

\( T(x, y) \leq \min(x, y) \) and \( \bot (x, y) \geq \max(x, y) \) for any \( T \) and \( \bot \)
The Special Role of Minimum and Maximum I

\( T_{\min} \) and \( \perp_{\max} \) play key role for intersection and union, resp. In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all \( x, y, z \in [0,1] \):

**Distributivity**
\[
\begin{align*}
\perp_{\max}(x, T_{\min}(y, z)) &= T_{\min}(\perp_{\max}(x, y), \perp_{\max}(x, z)), \\
T_{\min}(x, \perp_{\max}(y, z)) &= \perp_{\max}(T_{\min}(x, y), T_{\min}(x, z))
\end{align*}
\]

**Continuity**

\( T_{\min} \) and \( \perp_{\max} \) are continuous.
The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\text{min}}(x, x) < \top_{\text{min}}(y, y)$ and $\bot_{\text{max}}(x, x) < \bot_{\text{max}}(y, y)$.

Idempotency

$\top_{\text{min}}(x, x) = x$, $\bot_{\text{max}}(x, x) = x$

Absorption

$\top_{\text{min}}(x, \bot_{\text{max}}(x, y)) = x$, $\bot_{\text{max}}(x, \top_{\text{min}}(x, y)) = x$

Non-compensation

$x < y < z$ imply $\top_{\text{min}}(x, z) \neq \top_{\text{min}}(y, y)$ and $\bot_{\text{max}}(x, z) \neq \bot_{\text{max}}(y, y)$. 
The Minimum and Maximum II

$T_{\min}$ and $\perp_{\max}$ can be easily processed numerically and visually, e.g. linguistic values *young* and *approx. 20* described by $\mu_y, \mu_{20}$. $T_{\min}(\mu_y, \mu_{20})$ is shown below.
The Product and Probabilistic Sum

\[ T_{\text{prod}}(x, y) = x \cdot y, \quad \perp_{\text{sum}}(x, y) = x + y - x \cdot y \]
The Łukasiewicz $t$-norm and $t$-conorm

\[ T_{Łuka}(x, y) = \max\{0, x + y - 1\}, \quad \bot_{Łuka}(x, y) = \min\{1, x + y\} \]

$T_{Łuka}$, $\bot_{Łuka}$ are also called \textit{bold intersection} and \textit{boundedsum}.
### The Drastic Product and Sum

\[ T_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \bot_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases} \]

\( T_{-1} \) is the weakest \( t \)-norm, \( \bot_{-1} \) is the strongest \( t \)-conorm.

\( T_{-1} \leq T \leq T_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1} \) for any \( T \) and \( \bot \)
Examples of Fuzzy Intersections

Note that all fuzzy intersections are contained within upper left graph and lower right one.
Examples of Fuzzy Unions

- $t$-conorm $\perp_{\text{max}}$
- $t$-conorm $\perp_{\text{sum}}$
- $t$-conorm $\perp_{\text{luka}}$
- $t$-conorm $\perp_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.
Continuous Archimedian $t$-norms and $t$-conorms

Often it is possible to representation functions with several inputs by a function with only one input, e.g.

$$K(x, y) = f^{(-1)}(f(x) + f(y))$$

For a subclass of $t$-norms this is possible. The trick makes calculations simpler.

A $t$-norm $T$ is called
(a) \textit{continuous} if $T$ is continuous
(b) \textit{Archimedian} if $T$ is continuous and $T(x, x) < x$ for all $x \in ]0, 1[$.

A $t$-conorm $\bot$ is called
(a) \textit{continuous} if $\bot$ is continuous,
(b) \textit{Archimedian} if $\bot$ is continuous and $\bot(x, x) > x$ for all $x \in ]0, 1[$.
The concept of a pseudoinverse

**Definition**

Let $f : [a, b] \rightarrow [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to $f$ is the function $f^{(-1)} : [c, d] \rightarrow [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing}, \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$
The concept of a pseudoinverse

\[ f^{-1}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases} \]
Archimedian t-norms

**Theorem**

A t-norm \( T \) is Archimedian if and only if there exists a strictly decreasing and continuous function \( f : [0, 1] \to [0, \infty) \) with \( f(1) = 0 \) such that

\[
T(x, y) = f^{-1}(f(x) + f(y))
\]

where

\[
f^{-1}(x) = \begin{cases} 
  f^{-1}(x) & \text{if } x \leq f(0) \\
  0 & \text{otherwise}
\end{cases}
\]

is the pseudoinverse of \( f \). Moreover, this representation is unique up to a positive multiplicative constant.

\( T \) is generated by \( f \) if \( T \) has representation (1).

\( f \) is called additive generator of \( T \).
Additive Generators of $t$-norms – Examples

Find an additive generator $f$ of $T_{\text{Łuka}}(x, y) = \max\{x + y - 1, 0\}$.

For instance, $f_{\text{Łuka}}(x) = 1 - x$

Then, $f^{(-1)}_{\text{Łuka}}(x) = \max\{1 - x, 0\}$

Thus $T_{\text{Łuka}}(x, y) = f^{(-1)}_{\text{Łuka}}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator $f$ of $T_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

Hint: use of logarithmic and exponential function
Archimedian $t$-conorms

**Theorem**

A $t$-conorm $\perp$ is continuous and Archimedian if and only if there exists a strictly increasing and continuous function $g : [0, 1] \to [0, \infty]$ with $g(0) = 0$ such that

$$\perp(x, y) = g^{-1}(g(x) + g(y))$$

where

$$g^{-1}(x) = \begin{cases} 1 & \text{if } x \leq g(1) \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

is the pseudoinverse of $g$. Moreover, this representation is unique up to a positive multiplicative constant.

$\perp$ is generated by $g$ if $\perp$ has representation (2).

$g$ is called additive generator of $\perp$. 
Additive Generators of $t$-conorms – Two Examples

Find an additive generator $g$ of $\perp_{\text{Łuka}}(x, y) = \min\{x + y, 1\}$.

For instance $g_{\text{Łuka}}(x) = x$

then, $g_{\text{Łuka}}^{(-1)}(x) = \min\{x, 1\}$

thus $\perp_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator $g$ of $\perp_{\text{sum}}(x, y) = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.
Sugeno-Weber Family I

For $\lambda > -1$ and $x, y \in [0, 1]$, define

$$T_\lambda(x, y) = \max \left\{ \frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0 \right\},$$

$$\perp_\lambda(x, y) = \min \{ x + y + \lambda xy, 1 \}.$$

$\lambda = 0$ leads to $T_\text{Łuka}$ and $\perp_\text{Łuka}$, resp.

$\lambda \to \infty$ results in $T_{\text{prod}}$ and $\perp_{\text{sum}}$, resp.

$\lambda \to -1$ creates $T_{-1}$ and $\perp_{-1}$, resp.
Additive generators $f_\lambda$ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 
1 - x & \text{if } \lambda = 0 \\
1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.}
\end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter $\lambda$. Additive generators of $\perp_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$. 
warm and hot?
Zadeh's Intersection

\[ a \text{ and } b = \min (a, b), \text{ for all membership degrees } a, b \]

\[ (\mu_{\text{warm}} \cap \mu_{\text{hot}})(x) = \min(\mu_{\text{warm}}(x), \mu_{\text{hot}}(x)), \text{ for all real numbers } x \]
Fuzzy Sets Inclusion
Subset Property

For Classical Sets: $x \in A \Rightarrow x \in B,$

\[ A \subseteq B \]

\[ \text{not} \ (A \subseteq B) \]

For Fuzzy Sets: $x \in \mu \Rightarrow x \in \mu'$
Fuzzy Set Implication
How to model
if speed is fast then distance is high

A straightforward solution with a multivalued logic

- Define fuzzy sets for fast and high
- Determine for all speed values $x$ and all distance values $y$ the membership degrees (i.e. its truth value)
- Calculate for each pair $x$ and $y$ the truth value of the implication

$$ \mu_{\text{fast}}(x) \Rightarrow \mu_{\text{high}}(y) $$
Definition of a Multivalued Implication

1. One way of defining $I$ is to use the property that in classical logic the propositions $a \Rightarrow b$ and $\neg a \lor b$ have the same truth values for all truth assignments to $a$ and $b$.

If we model the disjunction and negation as $t$-conorm and fuzzy complement, resp., then for all $a, b \in [0,1]$ the following definition of a fuzzy implication seems reasonable:

$$I(a, b) = \bot(\sim a, b).$$

2. Another way is to use the concept of a residuum in classical logic: $a \Rightarrow b$ and $\max\{x \in \{0, 1\} \mid a \land x \leq b\}$ have the same truth values for all truth assignments for $a, b$. If in a generalized logic the conjunction is modelled by a $t$-norm, then a reasonable generalization could be:

$$I(a, b) = \sup\{x \in [0, 1] \mid T(a, x) \leq b\}.$$
Definition of a Multivalued Implication

3. Another proposal is to use the fact that, in classical logic, the propositions $a \Rightarrow b$ and $\neg a \lor (a \land b)$ have the same truth for all truth assignments.

A possible extension to many valued logics is therefore

$$I(a, b) = \bot(\neg a, T(a, b)),$$

where $(T, \bot, \neg)$ should be a *De Morgan triplet*.

So again, the classical definition of an implication is unique, whereas there is a „zoo“ of fuzzy implications.

Typical question for applications: *What to use when and why?*
Implications based on \( I(a, b) = \perp(\sim a, b) \) are called **S-implications**. Symbol \( S \) is often used to denote \( t \)-conorms.

Four well-known \( S \)-implications are based on \( \sim a = 1 - a \):

<table>
<thead>
<tr>
<th>Name</th>
<th>( I(a, b) )</th>
<th>( \perp(a, b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kleene-Dienes</td>
<td>( I_{\text{max}}(a, b) = \max(1 - a, b) )</td>
<td>( \max(a, b) )</td>
</tr>
<tr>
<td>Reichenbach</td>
<td>( I_{\text{sum}}(a, b) = 1 - a + ab )</td>
<td>( a + b - ab )</td>
</tr>
<tr>
<td>Łukasiewicz</td>
<td>( I_{\text{Ł}}(a, b) = \min(1, 1 - a + b) )</td>
<td>( \min(1, a + b) )</td>
</tr>
</tbody>
</table>
| **largest**       | \( I_{-1}(a, b) = \begin{cases} 
  b, & \text{if } a = 1 \\
  1 - a, & \text{if } b = 0 \\
  1, & \text{otherwise}
\end{cases} \) | \( \begin{cases} 
  b, & \text{if } a = 0 \\
  a, & \text{if } b = 0 \\
  1, & \text{otherwise}
\end{cases} \) |
$R$-Implications

$I(a, b) = \sup \{ x \in [0, 1] \mid T(a, x) \leq b \}$ leads to $R$-implications.

Symbol $R$ represents close connection to residuated semigroup.

Three well-known $R$-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to Gödel implication

  \[ I_{\min}(a, b) = \sup \{ x \mid \min(a, x) \leq b \} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases} \]

- Product leads to Goguen implication

  \[ I_{\text{prod}}(a, b) = \sup \{ x \mid ax \leq b \} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases} \]

- Łukasiewicz $t$-norm leads to Łukasiewicz implication

  \[ I_{\text{Ł}}(a, b) = \sup \{ x \mid \max(0, a + x - 1) \leq b \} = \min(1, 1 - a + b) \]
**QL-Implications**

Implications based on \( I(a, b) = \bot(\sim a, \top(a, b)) \) are called **QL-implications** (*QL* from quantum logic).

Four well-known *QL*-implications are based on \( \sim a = 1 - a \):

- **Zadeh implication**
  \[
  I_Z(a, b) = \max[1 - a, \min(a, b)].
  \]
- **Kleene-Dienes implication** again.
- **Using** \( \top \) and \( \bot \) leads to **Kleene-Dienes implication** again.
- Using \( \top \) and \( \bot \) leads to **Kleene-Dienes implication** again.

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All $I$ come from generalizations of the classical implication. They collapse to the classical implication when truth values are 0 or 1. Generalizing classical properties leads to the following propositions:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$  \hfill (monotonicity in 1st argument)
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$  \hfill (monotonicity in 2nd argument)
3) $I(0, a) = 1$  \hfill (dominance of falsity)
4) $I(1, b) = b$  \hfill (neutrality of truth)
5) $I(a, a) = 1$  \hfill (identity)
6) $I(a, I(b, c)) = I(b, I(a, c))$  \hfill (exchange property)
7) $I(a, b) = 1$ if and only if $a \leq b$  \hfill (boundary condition)
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$  \hfill (contraposition)
9) $I$ is a continuous function  \hfill (continuity)
Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem
A function \( I : [0, 1]^2 \rightarrow [0, 1] \) satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement \( \sim \) if and only if there exists a strict increasing continuous function \( f : [0, 1] \rightarrow [0, \infty) \) such that \( f(0) = 0 \),

\[
I(a, b) = f^{-1}(f(1) - f(a) + f(b))
\]

for all \( a, b \in [0, 1] \), and

\[
\sim a = f^{-1}(f(1) - f(a))
\]

for all \( a \in [0, 1] \).
Example

Consider $f_{\lambda}(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$. Its pseudo-inverse is

$$f_{\lambda}^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy negation generated by $f_{\lambda}$ for all $a \in [0, 1]$ is

$$n_{\lambda}(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_{\lambda}(a, b) = \min \left( 1, \frac{1 - a + b + \lambda b}{1 + \lambda a} \right).$$

If $\lambda \in (-1, 0)$, then $I_{\lambda}$ is called pseudo-Łukasiewicz implication.