Propagation in Clique Trees

Potential Representation

Let $V = \{X_j\}$ be a set of random variables $X_j : \Omega \to \mathrm{dom}(X_j)$ and P the joint distribution over V. Further, let

$$\{W_i \mid W_i \subseteq V, 1 \leq i \leq p\}$$

a family of subsets of V with associated functions

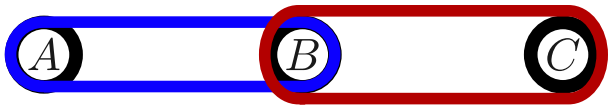
$$\psi_i : \bigtimes_{X_j \in W_i} \mathrm{dom}(X_j) \to \mathbb{R}$$

It is said that $P(V)$ *factorizes* according $\Big(\{W_1, \ldots, W_p\}, \{\psi_1, \ldots, \psi_p\}\Big)$ if $P(V)$ can be written as:

$$P(v) = k \cdot \prod_{i=1}^{p} \psi_i(w_i)$$

where $k \in \mathbb{R}$, w_i is a realization of W_i that meets the values of v.

Example



$$V = \{A, B, C\},\ W_1 = \{A, B\},\ W_2 = \{B, C\}$$

$$\begin{aligned}\mathrm{dom}(A) &= \{a_1, a_2\} \\ \mathrm{dom}(B) &= \{b_1, b_2\} \\ \mathrm{dom}(C) &= \{c_1, c_2\}\end{aligned}$$

$$P(a, b, c) = \tfrac{1}{8}$$

$$\begin{aligned}\psi_1 &: \{a_1, a_2\} \times \{b_1, b_2\} \to \mathbb{R} \\ \psi_2 &: \{b_1, b_2\} \times \{c_1, c_2\} \to \mathbb{R}\end{aligned}$$

$$\begin{aligned}\psi_1(a, b) &= \tfrac{1}{4} \\ \psi_2(b, c) &= \tfrac{1}{2}\end{aligned}$$

$(\{W_1, W_2\}, \{\psi_1, \psi_2\})$ is a potential representation of P.

Factorization of a Belief Network

Let (V, E, P) be a belief network and $\{C_1, \ldots, C_p\}$ the cliques of the join tree. For every node $v \in V$ choose a clique C such that v and all of its parents are contained in C, i. e. $\{v\} \cup c(v) \subseteq C$. The chosen clique is designated as $f(v)$.

To arrive at a factorization $(\{C_1, \ldots, C_p\}, \{\psi_1, \ldots, \psi_p\})$ of P the factor potentials are:

$$\psi_i(c_i) = \prod_{v : f(v) = C_i} P(v \mid c(v))$$

Separator Sets and Residual Sets

Let $\{C_1, \ldots, C_p\}$ be a set of cliques w. r. t. V. The sets

$$S_i = C_i \cap (C_1 \cup \cdots \cup C_{i-1}), \qquad i = 2, \ldots, p, \qquad S_1 = \emptyset$$

are called *separator sets* with their corresponding *residual sets*

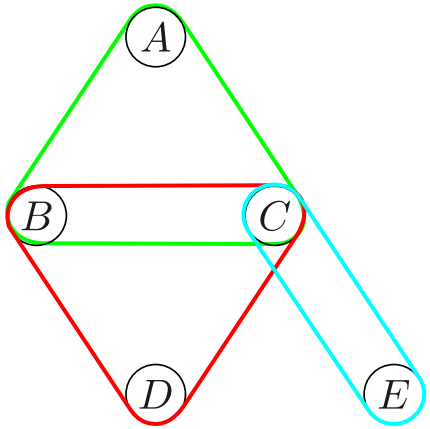
$$R_i = C_i \backslash S_i$$

Decomposition w. r. t. a Join-Tree

Given a clique ordering $\{C_1, \ldots, C_p\}$ that satisfies the RIP, we can conclude the following separation statements:

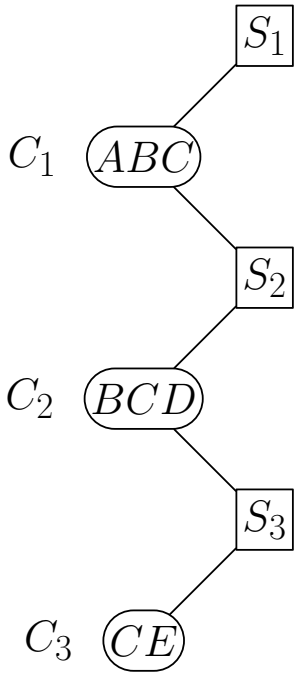
$$R_i \perp\!\!\!\perp (C_1 \cup \cdots \cup C_{i-1}) \backslash S_i \mid S_i \qquad \text{for } i > 1$$

Example



$$S_1 = \emptyset \qquad R_1 = \{A, B, C\}$$
$$S_2 = \{B, C\} \qquad R_2 = \{D\}$$
$$S_3 = \{C\} \qquad R_3 = \{E\}$$

$$f(A) = C_1$$
$$f(B) = C_1$$
$$f(C) = C_1$$
$$f(D) = C_2$$
$$f(E) = C_3$$



$$\psi_1(C_1) = P(A) \cdot P(C \mid A) \cdot P(B|A)$$
$$\psi_2(C_2) = P(D \mid B, C)$$
$$\psi_3(C_3) = P(E \mid C)$$

Propagation is accomplished by sending messages across the cliques in the tree. The emerging potentials are maintained by each clique.

Propagation in Join Trees

Main Idea

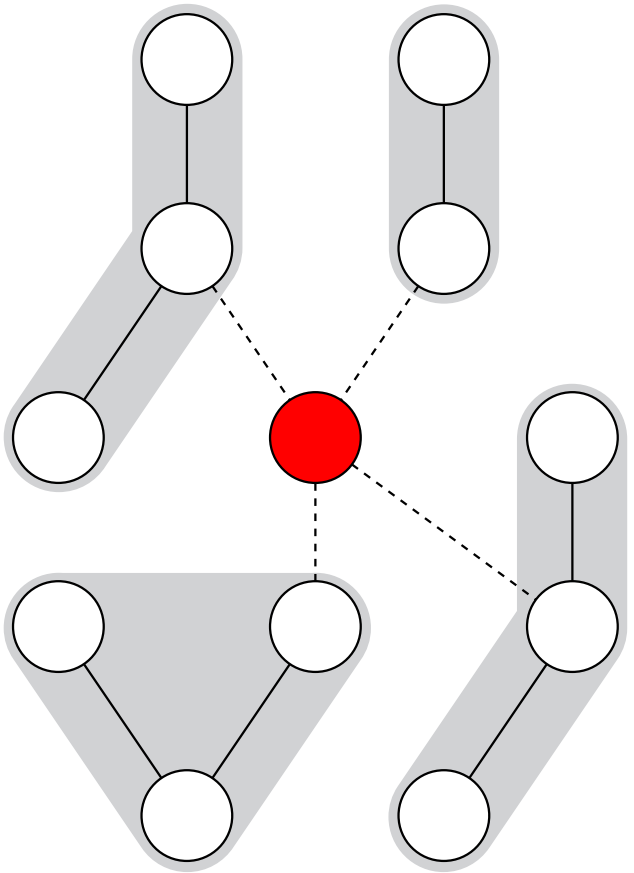
Incorporate evidence into the clique potentials.

Since we are dealing with a tree structure, exploit the fact that a clique "separates" all its neighboring cliques (and their respective subtrees) from each other.

Apply a message passing scheme to inform neighboring cliques about evidence.

Since we do not have edge directions, we will only need one type of message.

After having updated all cliques' potentials, we marginalize (and normalize) to get the probabilities of single attributes.



Incorporating Evidence

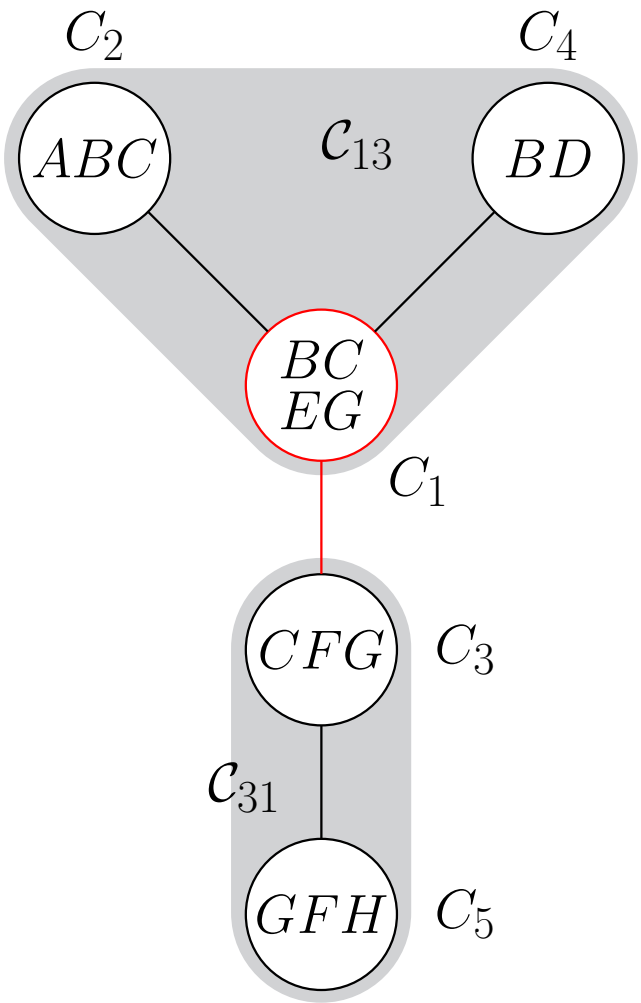
Every clique C_i maintains a potential function ψ_i.

If for an attribute E some evidence e becomes known, we alter all potential functions of cliques containing E as follows:

$$\psi_i^*(c_i) = \begin{cases} 0, & \text{if a value in } c_i \text{ is inconsistent with } e \\ \psi_i(c_i), & \text{otherwise} \end{cases}$$

All other potential functions are unchanged.

Notation and Nomenclature



In general:

Clique C_i has q neighboring cliques $B_1, \ldots, B_q$.

$\mathcal{C}_{ij}$ is the set of cliques in the subtree containing C_i after dropping the link to B_j.

X_{ij} is the set of attributes in the cliques of $\mathcal{C}_{ij}$.

$V = X_{ij} \cup X_{ji}$ (complementary sets)

$S_{ij} = S_{ji} = C_i \cap C_j$ (not shown here)

$R_{ij} = X_{ij} \setminus S_{ij}$ (not shown here)

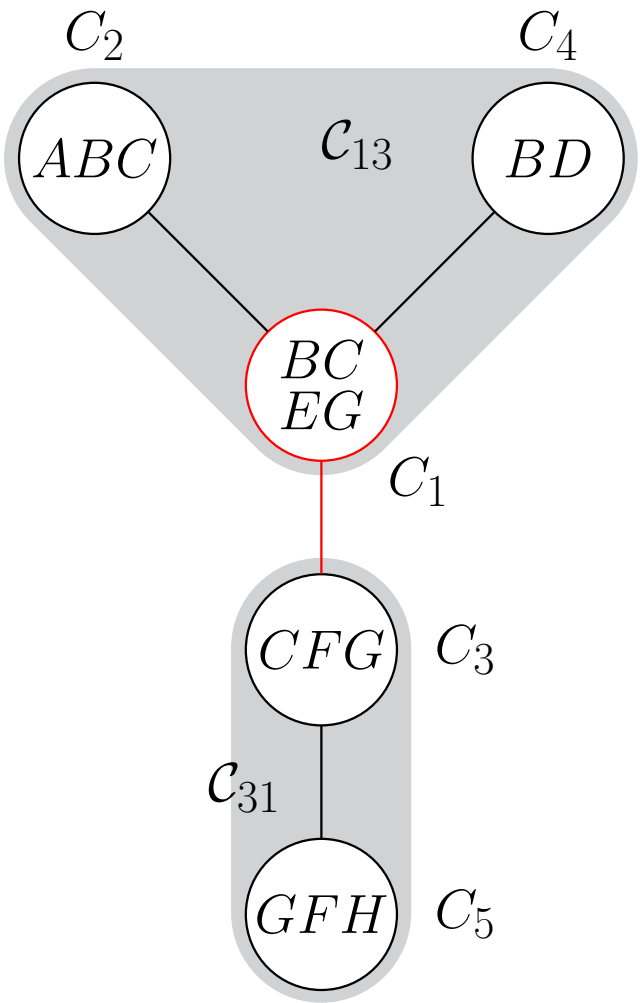
Here:

Neighbors of C_1: $\{C_2, C_4, C_3\}$, $\mathcal{C}_{13} = \{C_1, C_2, C_4\}$

$X_{13} = \{A, B, C, D, E, G\}$, $S_{13} = \{C, G\}$

$V = X_{13} \cup X_{31} = \{A, B, C, D, E, F, G, H\}$

$R_{13} = \{A, B, D, E\}$, $R_{31} = \{F, H\}$

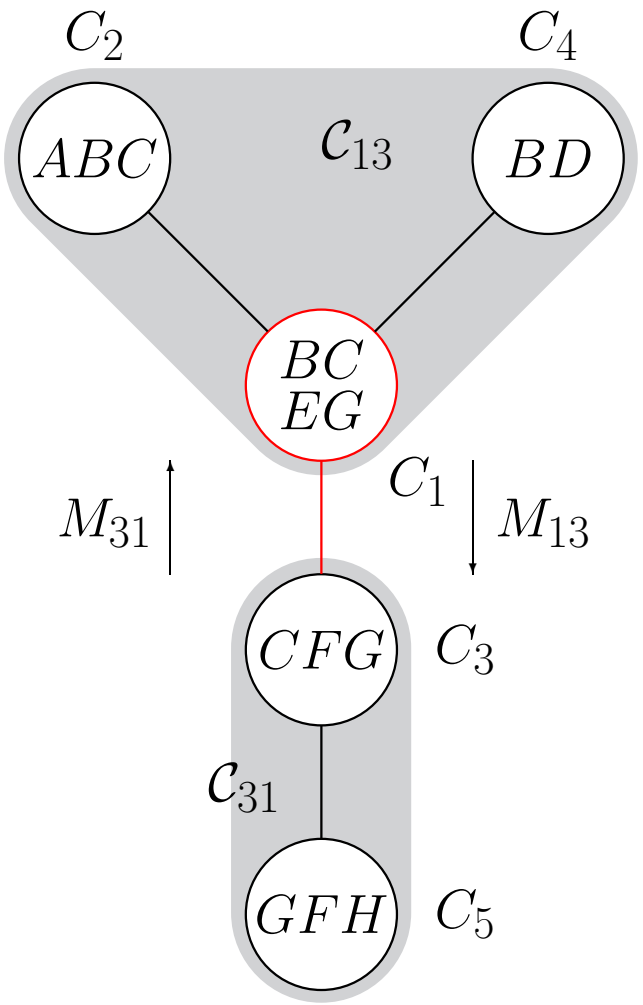


Task: Calculate $P(s_{ij})$:

$$\begin{aligned} V \setminus S_{ij} &= (X_{ij} \cup X_{ji}) \setminus S_{ij} \\ &= (X_{ij} \setminus S_{ij}) \cup (X_{ji} \setminus S_{ij}) \\ &= R_{ij} \cup R_{ji} \end{aligned}$$

$$\begin{aligned} V \setminus S_{13} &= (X_{13} \cup X_{31}) \setminus S_{13} \\ &= R_{13} \cup R_{31} \\ V \setminus \{C, G\} &= \{A, B, D, E\} \cup \{F, H\} \\ &= \{A, B, D, E, F, H\} \end{aligned}$$

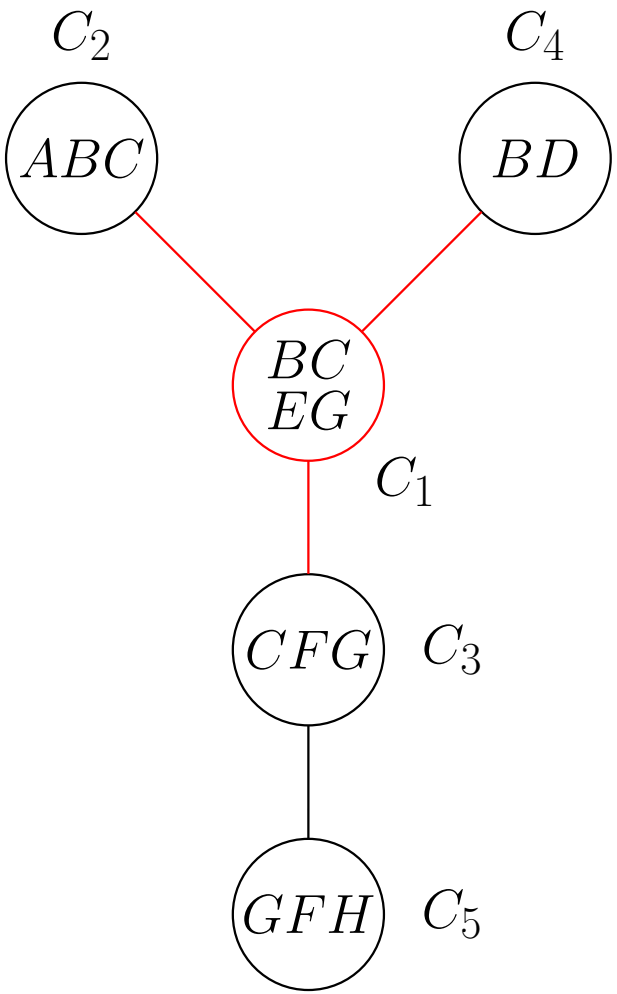
Note: R_{ij} is the set of attributes that are in C_i's subtree but not in B_j's. Therefore, R_{ij} and R_{ji} are always **disjoint**.



Task: Calculate $P(s_{ij})$:

$$
\begin{aligned}
P(s_{ij}) &= \sum_{v \setminus s_{ij}} \prod_{k=1}^{m} \psi_k(c_k) \\
&\overset{\text{last slide}}{=} \sum_{r_{ij} \cup r_{ji}} \prod_{k=1}^{m} \psi_k(c_k) \\
&\overset{\text{sum rule}}{=} \left(\sum_{r_{ij}} \prod_{c_k \in \mathcal{C}_{ij}} \psi_k(c_k) \right) \cdot \left(\sum_{r_{ji}} \prod_{c_k \in \mathcal{C}_{ji}} \psi_k(c_k) \right) \\
&= M_{ij}(s_{ij}) \cdot M_{ji}(s_{ij})
\end{aligned}
$$

M_{ij} is the message sent from C_i to neighbor B_j and vice versa.

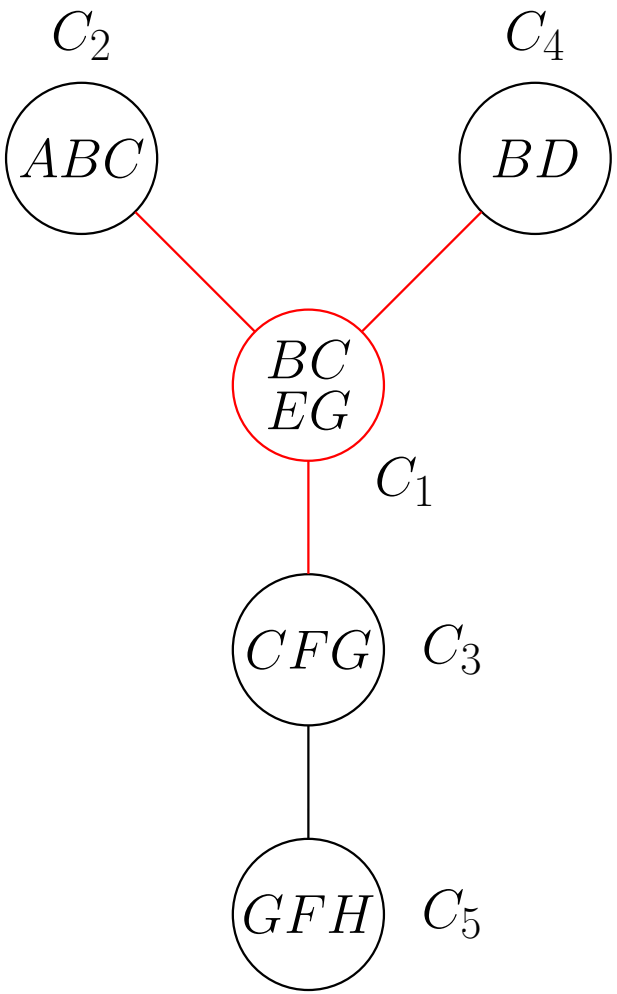


Task: Calculate $P(c_i)$:

$$\begin{aligned} V \setminus C_i &= \left(\bigcup_{k=1}^{q} X_{ki} \right) \setminus C_i \\ &= \bigcup_{k=1}^{q} \left(X_{ki} \setminus C_i \right) \\ &= \bigcup_{k=1}^{q} R_{ki} \end{aligned}$$

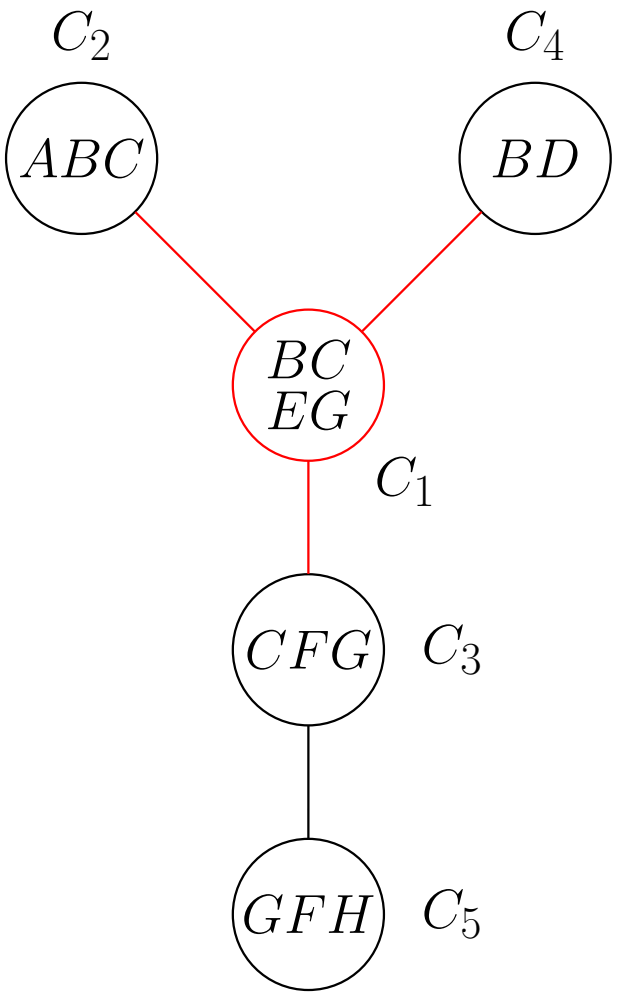
Example:

$$\begin{aligned} V \setminus C_1 &= R_{21} \cup R_{41} \cup R_{31} \\ \{A, D, F, H\} &= \{A\} \cup \{D\} \cup \{F, H\} \end{aligned}$$



Task: Calculate $P(c_i)$:

$$
\begin{aligned}
P(c_i) &= \underbrace{\sum_{v \setminus c_i}}_{\text{Marginalization}} \underbrace{\prod_{j=1}^{m} \psi_j(c_j)}_{\text{Decomposition}} \\
&= \psi_i(c_i) \sum_{v \setminus c_i} \prod_{i \neq j} \psi_j(c_j) \\
&= \psi_i(c_i) \sum_{r_{1i} \cup \cdots \cup r_{qi}} \prod_{i \neq j} \psi_j(c_j) \\
&= \psi_i(c_i) \underbrace{\left(\sum_{r_{1i}} \prod_{c_k \in \mathcal{C}_{1i}} \psi_k(c_k) \right)}_{M_{1i}(s_{ij})} \cdots \underbrace{\left(\sum_{r_{qi}} \prod_{c_k \in \mathcal{C}_{qi}} \psi_k(c_k) \right)}_{M_{qi}(s_{ij})} \\
&= \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})
\end{aligned}
$$



Example: $P(c_1)$:

$$P(c_1) = \psi_1(c_1) M_{21}(s_{12}) M_{41}(s_{14}) M_{31}(s_{13})$$

$M_{ij}(s_{ij})$ can be simplified further (without proof):

$$\begin{aligned} M_{ij}(s_{ij}) &= \sum_{r_{ij}} \prod_{c_k \in \mathcal{C}_{ij}} \psi_k(c_k) \\ &= \sum_{c_i \setminus s_{ij}} \psi_i(c_i) \prod_{k \neq j} M_{ki}(s_{ki}) \end{aligned}$$

Final Algorithm

Input: Join tree $(\mathcal{C}, \Psi)$ over set of variables V and evidence $E = e$.

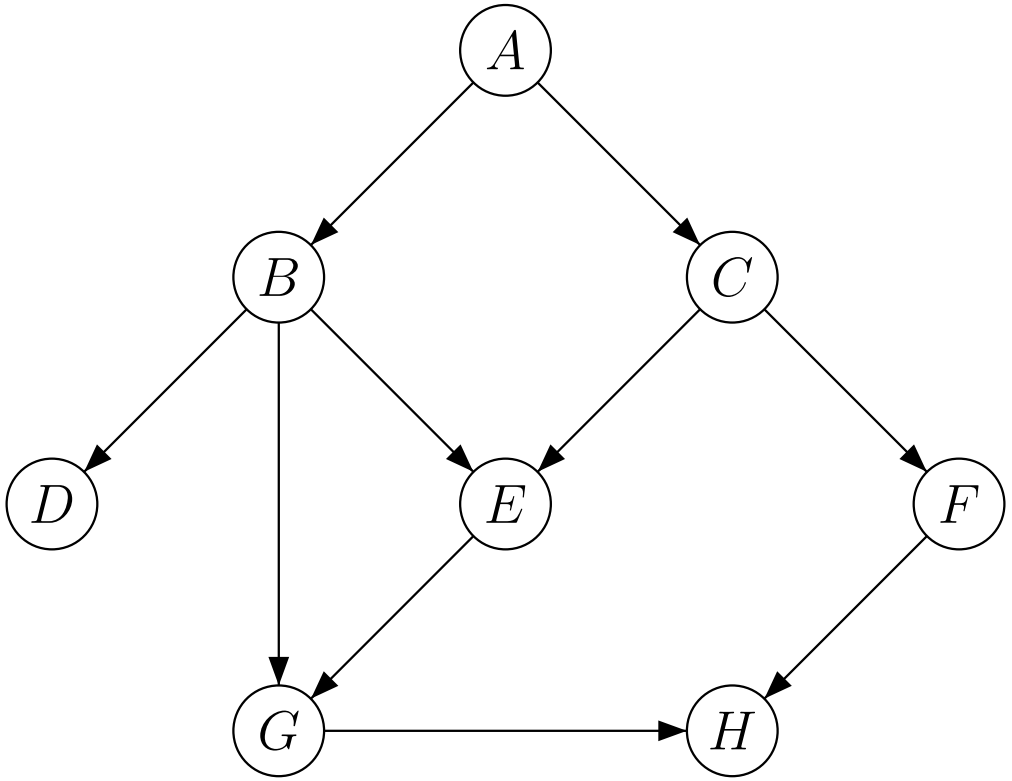
Output: The a-posteriori probability $P(x_i \,|\, e)$ for every non-evidential X_i.

Initialization: Incorporate evidence $E = e$ into potential functions.

Iterations:

1. For every clique C_i do: For every neighbor B_j of C_i do: If C_i has received all messages from the *other* neighbors, calculate and send $M_{ij}(s_{ij})$ to B_j.
2. Repeat step 1 until no message is calculated.
3. Calculate the joint probability distribution for every clique:
$$P(c_i) \;\propto\; \psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$
4. For every $X \in V$ calculate the a-posteriori probability:
$$P(x_i \mid e) \;=\; \sum_{c_k \setminus x_i} P(c_k)$$
where C_k is the smallest clique containing X_i.

Example: Putting it together

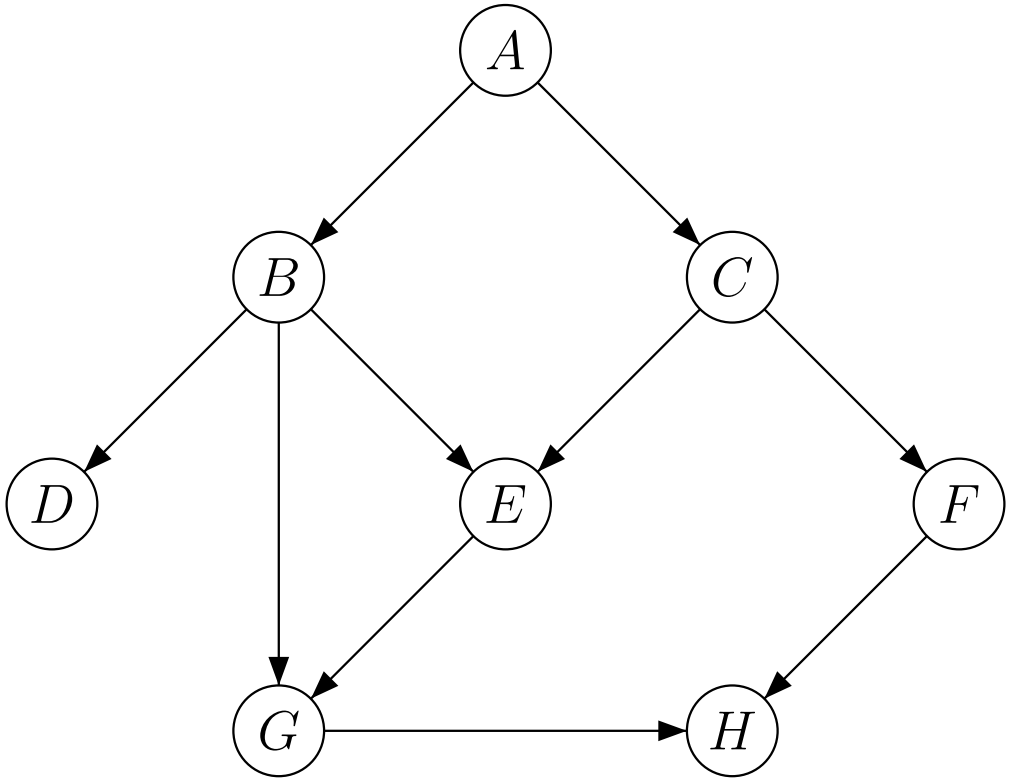


Goals: Find the marginal distributions and update them when evidence $H = h_1$ becomes known.

Steps:

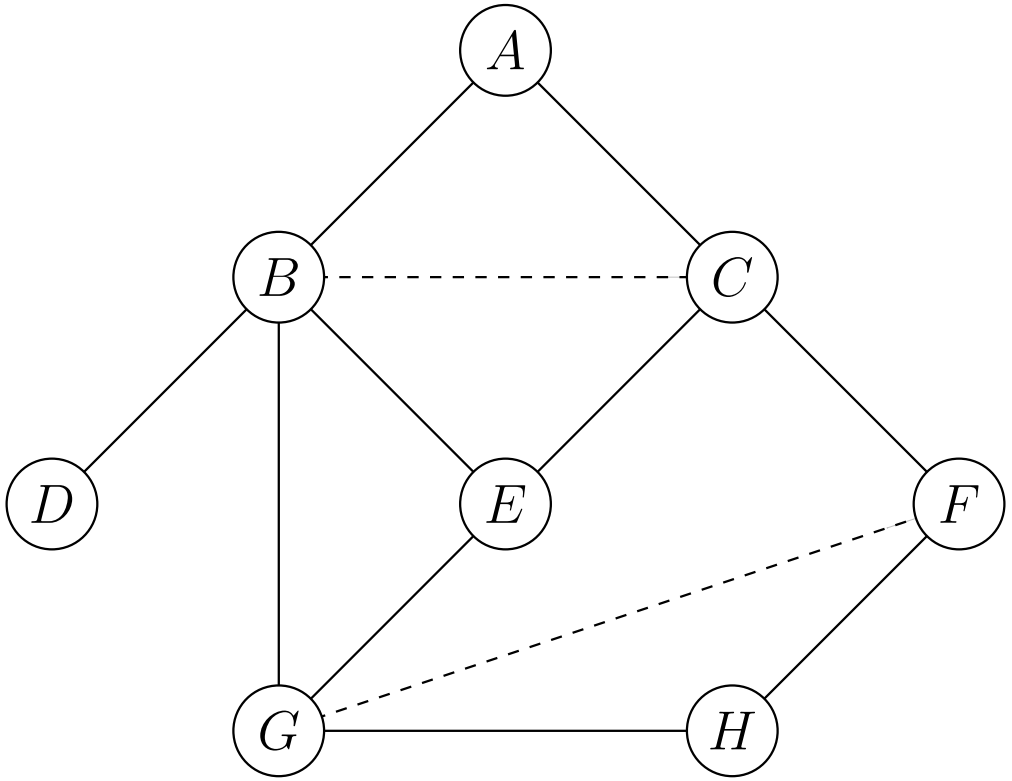
1. Transform network into join-tree.
2. Specify factor potentials.
3. Propagate "zero" evidence to obtain the marginals before evidence is present.
4. Update factor potentials w. r. t. the evidence and do another propagation run.

Example: Step 1: Find a Join-Tree



Join-Tree creation:

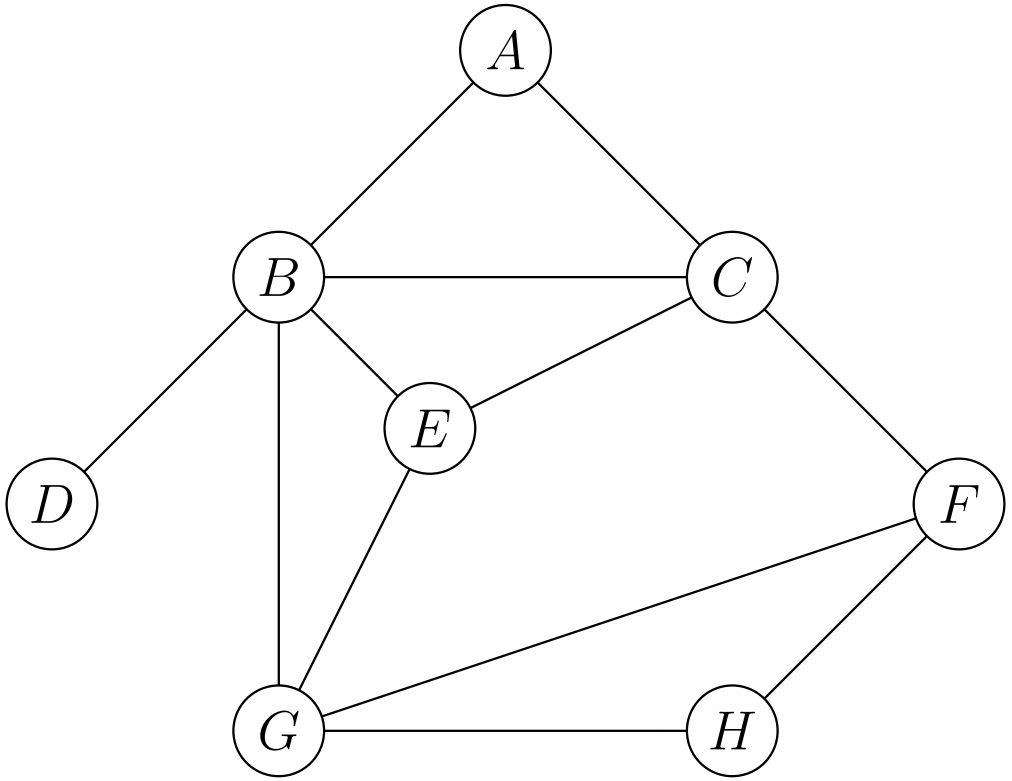
Example: Step 1: Find a Join-Tree



Join-Tree creation:

1. Moralize the graph.

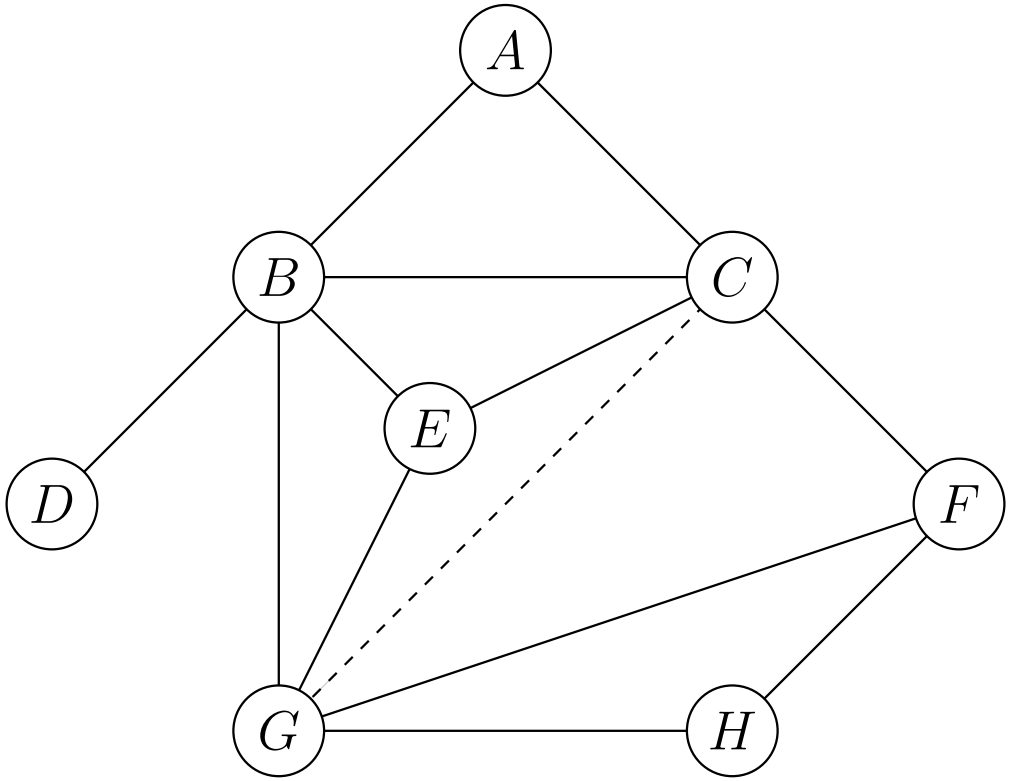
Example: Step 1: Find a Join-Tree



Join-Tree creation:

1. Moralize the graph.
2. Not yet triangulated.

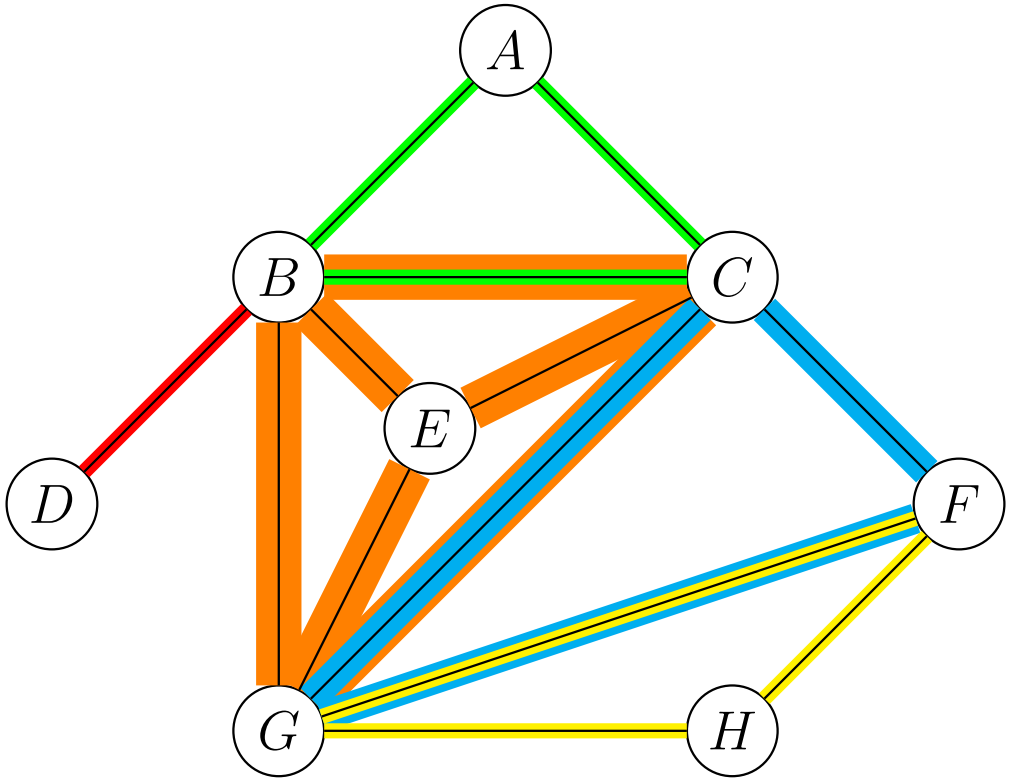
Example: Step 1: Find a Join-Tree



Join-Tree creation:

1. Moralize the graph.
2. Triangulate the graph.

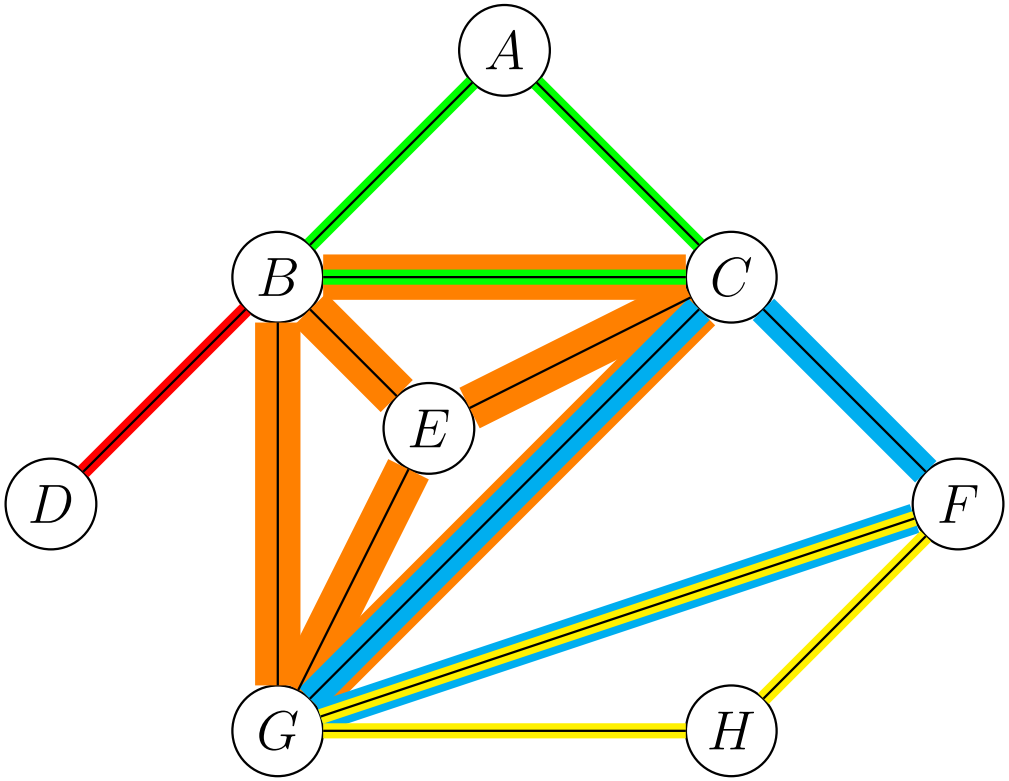
Example: Step 1: Find a Join-Tree



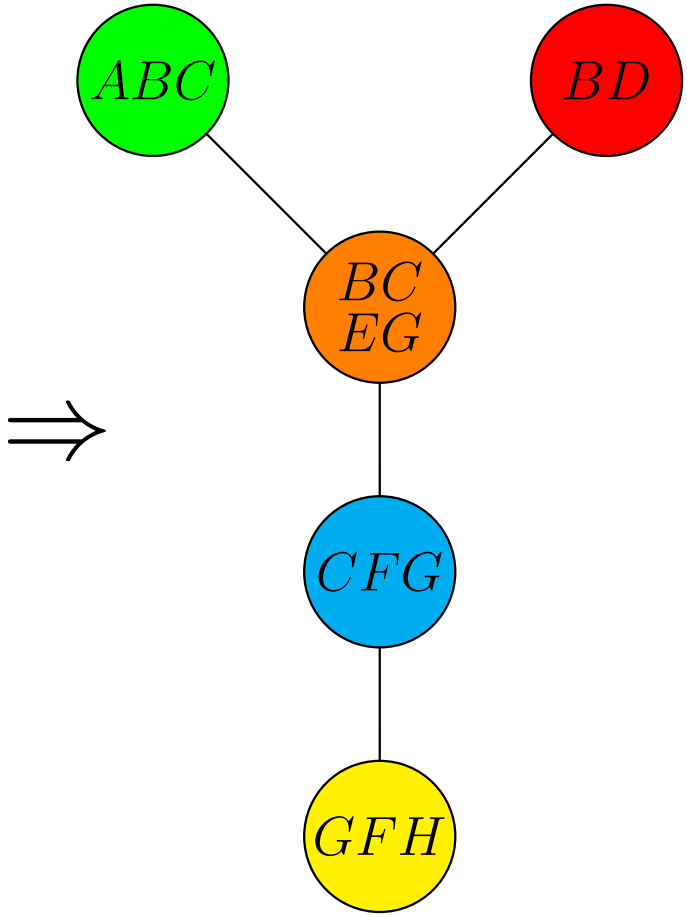
Join-Tree creation:

1. Moralize the graph.
2. Triangulate the graph.
3. Identify the maximal cliques.

Example: Step 1: Find a Join-Tree

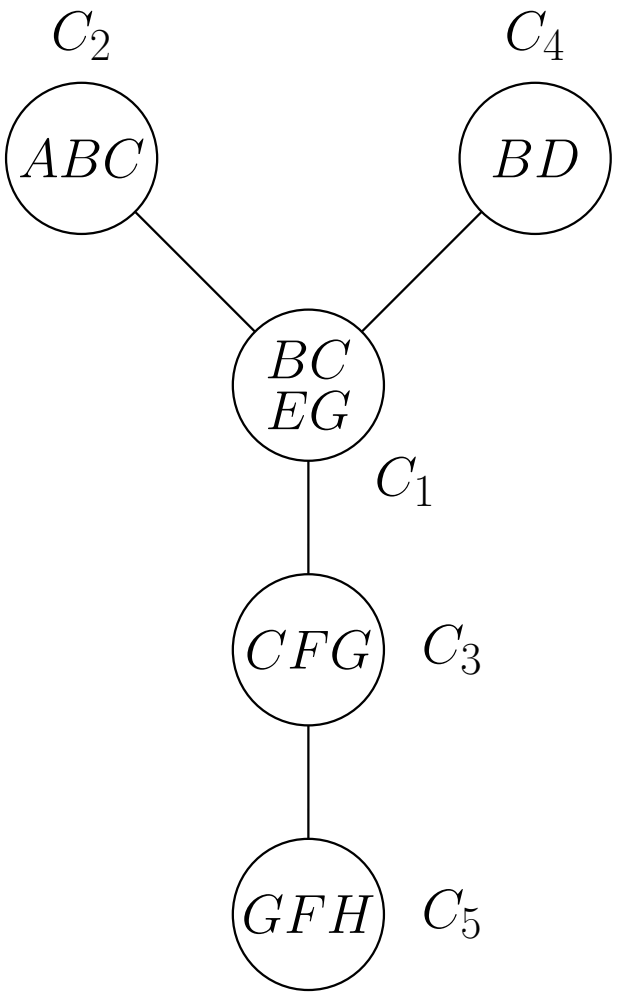


Example Bayesian network



One of the join trees

Example: Step 2: Specify the Factor Potentials



Decomposition of $P(A, B, C, D, E, F, G, H)$:

$$\begin{aligned} P(a, b, c, d, e, f, g, h) &= \prod_{i=1}^{5} \Psi_i(c_i) \\ &= \Psi_1(b, c, e, g) \cdot \Psi_2(a, b, c) \\ &\quad \cdot \Psi_3(c, f, g) \cdot \Psi_4(b, d) \\ &\quad \cdot \Psi_5(g, f, h) \end{aligned}$$

Where to get the factor potentials from?

Example: Step 2: Specify the Factor Potentials

As long as the factor potentials multiply together as on the previous slide, we are free to choose them.

Option 1: A factor potential of clique C_i is the product of all conditional probabilities of all node families properly contained in C_i:

$$\Psi_i(c_i) = 1 \cdot \prod_{\substack{\{X_i\} \cup Y_i \subseteq C_i \wedge \\ \text{parents}(X_i) = Y_i}} P(x_i \mid y_i)$$

The 1 stresses that if no node family satisfies the product condition, we assign a constant 1 to the potential.

Option 2: Choose potentials from the decomposition formula:

$$P(\bigcup_{i=1}^{n} C_i) = \frac{\prod_{i=1}^{n} P(C_i)}{\prod_{j=1}^{m} P(S_j)}$$

Example: Step 2: Specify the Factor Potentials

Option 1: Factor potentials according to the conditional distributions of the node families of the underlying Bayesian network:

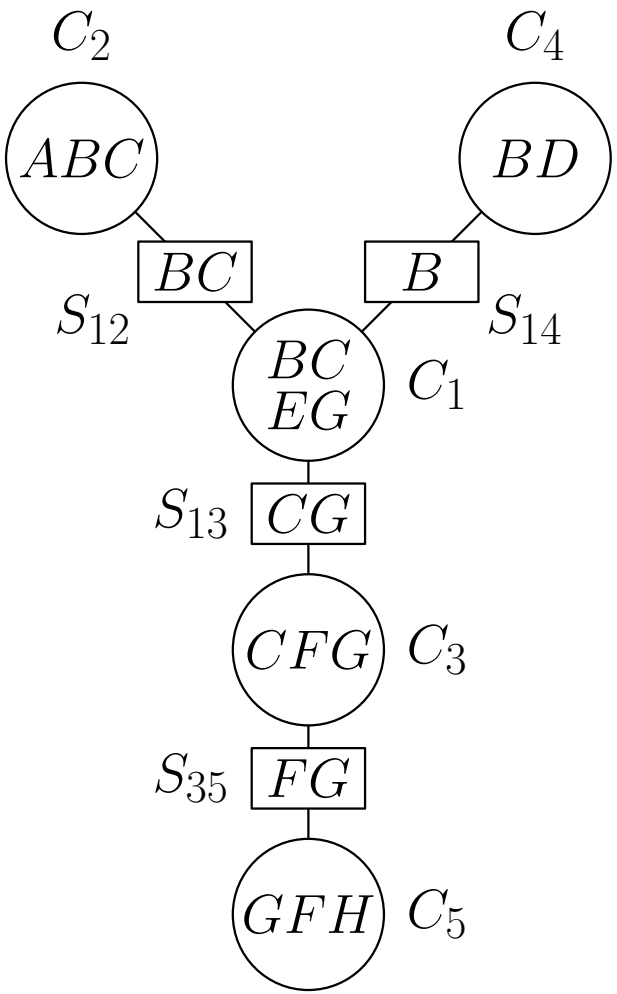
$$\begin{aligned}
\Psi_1(b, c, e, g) &= P(e \mid b, c) \cdot P(g \mid e, b) \\
\Psi_2(a, b, c) &= P(b \mid a) \cdot P(c \mid a) \cdot P(a) \\
\Psi_3(c, f, g) &= P(f \mid c) \\
\Psi_4(b, d) &= P(d \mid b) \\
\Psi_5(g, f, h) &= P(h \mid g, f)
\end{aligned}$$

(This assignment of factor potentials is used in this example.)

Option 2: Factor potentials chosen from the join-tree decomposition:

$$\begin{aligned}
\Psi_1(b, c, e, g) &= P(b, e \mid c, g) \\
\Psi_2(a, b, c) &= P(a \mid b, c) \\
\Psi_3(c, f, g) &= P(c \mid f, g) \\
\Psi_4(b, d) &= P(d \mid b) \\
\Psi_5(g, f, h) &= P(h, g, f)
\end{aligned}$$

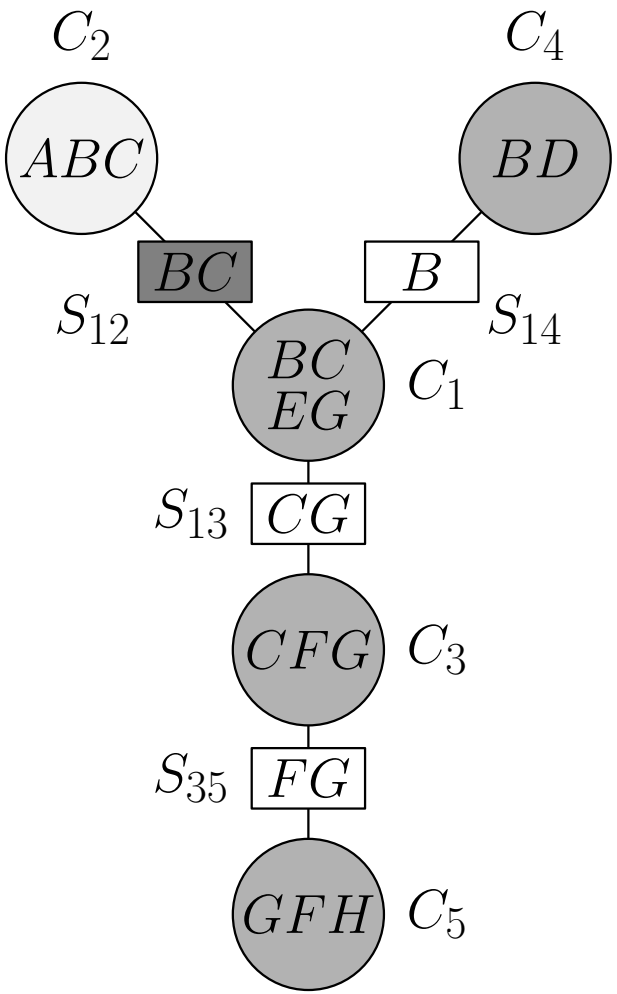
Example: Closer Look on Option 2: Separation in a Join-Tree



Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

Example: Closer Look on Option 2: Separation in a Join-Tree

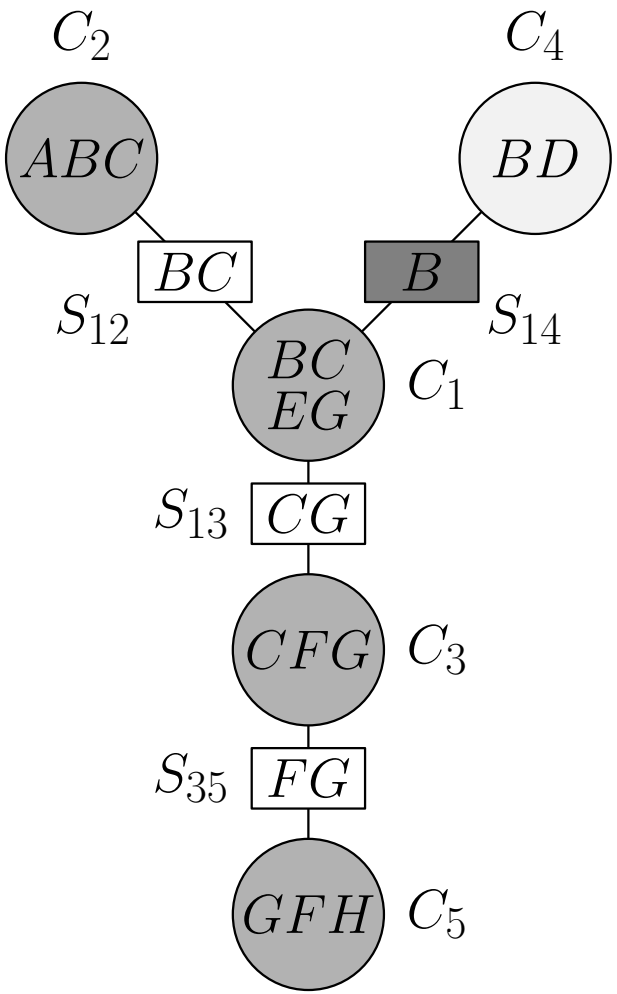


Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$

Example: Closer Look on Option 2: Separation in a Join-Tree

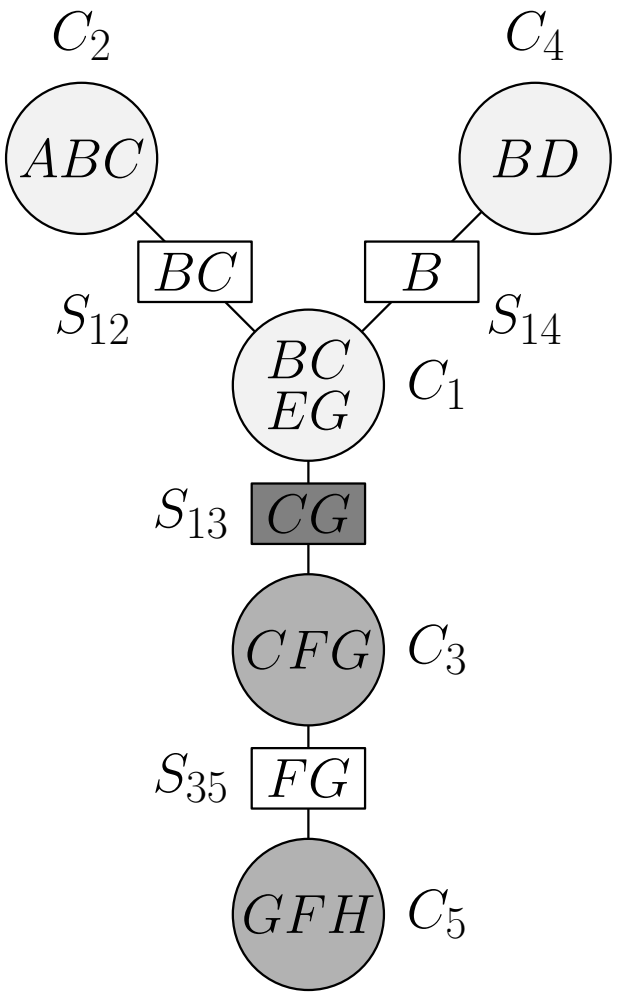


Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$

Example: Closer Look on Option 2: Separation in a Join-Tree

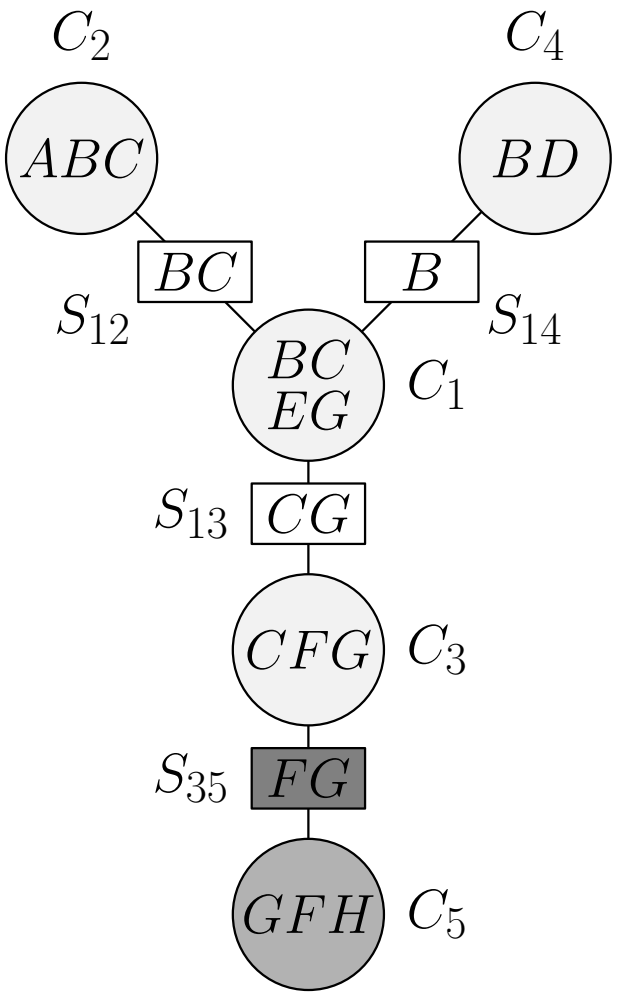


Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$
$$A, B, E, D \perp\!\!\!\perp F, H \mid G, C$$

Example: Closer Look on Option 2: Separation in a Join-Tree



Encoded independence statements:

Given any separator, the variables in the cliques on one side become independent of the variables in the cliques on the other side.

$$A \perp\!\!\!\perp D, E, F, G, H \mid B, C$$
$$D \perp\!\!\!\perp A, C, E, F, G, H \mid B$$
$$A, B, E, D \perp\!\!\!\perp F, H \mid G, C$$
$$H \perp\!\!\!\perp A, B, C, D, E \mid F, G$$

Example: Closer Look on Option 2: Decomposition

The four separation statements translate into the following independence statements:

$$
\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &&= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B &\Rightarrow P(D \mid B, C, E, F, G, H) &&= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C &\Rightarrow P(B, E \mid G, C, F, H) &&= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G &\Rightarrow P(C \mid F, G, H) &&= P(C \mid F, G)
\end{aligned}
$$

According to the chain rule we always have the following relation:

$$
\begin{aligned}
P(A, B, C, D, E, F, G, H) = {} & P(A \mid B, C, D, E, F, G, H) \cdot \\
& P(D \mid B, C, E, F, G, H) \cdot \\
& P(B, E \mid C, F, G, H) \cdot \\
& P(C \mid F, G, H) \cdot \\
& P(F, G, H)
\end{aligned}
$$

Example: Closer Look on Option 2: Decomposition

The four separation statements translate into the following independence statements:

$$
\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &&= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B &\Rightarrow P(D \mid B, C, E, F, G, H) &&= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C &\Rightarrow P(B, E \mid G, C, F, H) &&= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G &\Rightarrow P(C \mid F, G, H) &&= P(C \mid F, G)
\end{aligned}
$$

Exploiting the above independencies yields:

$$
\begin{aligned}
P(A, B, C, D, E, F, G, H) = {} & P(A \mid B, C) \cdot \\
& P(D \mid B) \cdot \\
& P(B, E \mid C, G) \cdot \\
& P(C \mid F, G) \cdot \\
& P(F, G, H)
\end{aligned}
$$

Example: Closer Look on Option 2: Decomposition

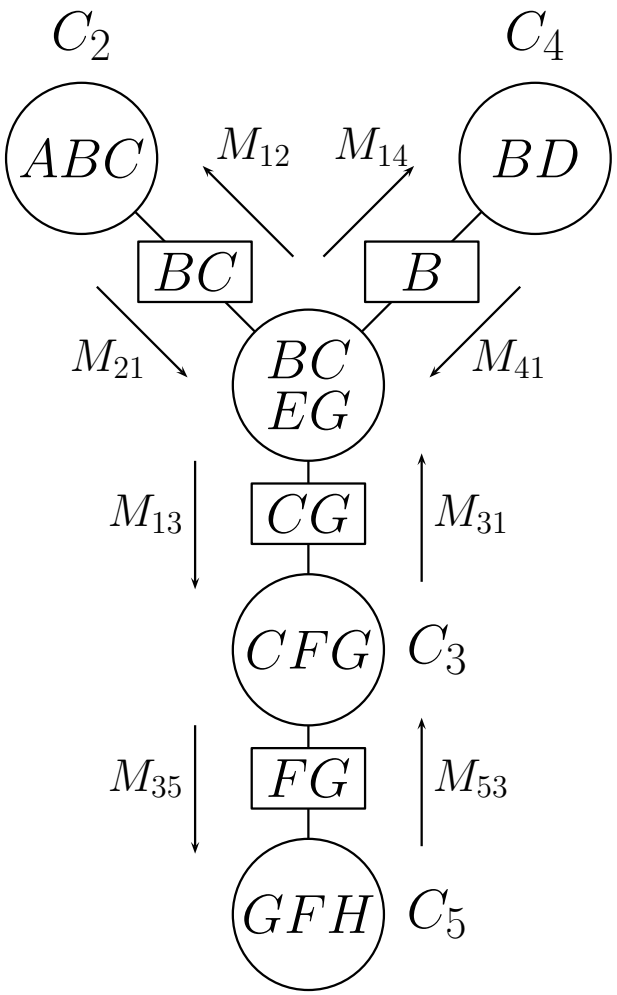
The four separation statements translate into the following independence statements:

$$
\begin{aligned}
A \perp\!\!\!\perp D, E, F, G, H \mid B, C &\Leftrightarrow P(A \mid B, C, D, E, F, G, H) &&= P(A \mid B, C) \\
D \perp\!\!\!\perp A, C, E, F, G, H \mid B &\Rightarrow P(D \mid B, C, E, F, G, H) &&= P(D \mid B) \\
A, B, E, D \perp\!\!\!\perp F, H \mid G, C &\Rightarrow P(B, E \mid G, C, F, H) &&= P(B, E \mid G, C) \\
H \perp\!\!\!\perp A, B, C, D, E \mid F, G &\Rightarrow P(C \mid F, G, H) &&= P(C \mid F, G)
\end{aligned}
$$

Getting rid of the conditions results in the final decomposition equation:

$$
\begin{aligned}
P(A, B, C, D, E, F, G, H) &= P(A \mid B, C) P(D \mid B) P(B, E \mid C, G) P(C \mid F, G) P(F, G, H) \\
&= \frac{P(A, B, C) P(D, B) P(B, E, C, G) P(C, F, G) P(F, G, H)}{P(B, C) P(B) P(C, G) P(F, G)} \\
&= \frac{P(C_1) P(C_2) P(C_3) P(C_4) P(C_5)}{P(S_{12}) P(S_{14}) P(S_{13}) P(S_{35})}
\end{aligned}
$$

Example: Step 3: Messages to be sent for Propagation



According to the join-tree propagation algorithm, the probability distributions of all clique instantiations c_i is calculated as follows:

$$P(c_i) \propto \Psi_i(c_i) \prod_{j=1}^{q} M_{ji}(s_{ij})$$

Spelt out for our example, we get:

$$\begin{aligned}
P(c_1) = P(b,c,e,g) &= \Psi_1(b,c,e,g) \cdot M_{21}(b,c) \cdot M_{31}(c,g) \cdot M_{41}(b) \\
P(c_2) = P(a,b,c) &\propto \Psi_2(a,b,c) \cdot M_{12}(b,c) \\
P(c_3) = P(c,f,g) &\propto \Psi_3(c,f,g) \cdot M_{13}(c,g) \cdot M_{53}(f,g) \\
P(c_4) = P(b,d) &\propto \Psi_4(b,d) \cdot M_{14}(b) \\
P(c_5) = P(f,g,h) &\propto \Psi_5(f,g,h) \cdot M_{35}(f,g)
\end{aligned}$$

The $\propto$-symbol indicates that the right-hand side may not add up to one. In that case we just normalize.

Example: Step 3: Message Computation Order

The structure of the join-tree imposes a partial ordering according to which the messages need to be computed:

$$M_{41}(b) = \sum_{d} \Psi_4(b, d)$$

$$M_{53}(f, g) = \sum_{h} \Psi_5(f, g, h)$$

$$M_{21}(b, c) = \sum_{a} \Psi_2(a, b, c)$$

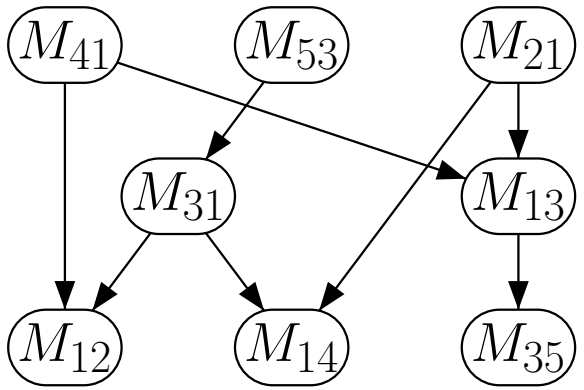
$$M_{31}(c, g) = \sum_{f} \Psi_3(c, f, g) M_{53}(f, g)$$

$$M_{13}(c, g) = \sum_{b,e} \Psi_1(b, c, e, g) M_{21}(b, c) M_{41}(b)$$

$$M_{12}(b, c) = \sum_{e,g} \Psi_2(b, c, e, g) M_{31}(c, g) M_{41}(b)$$

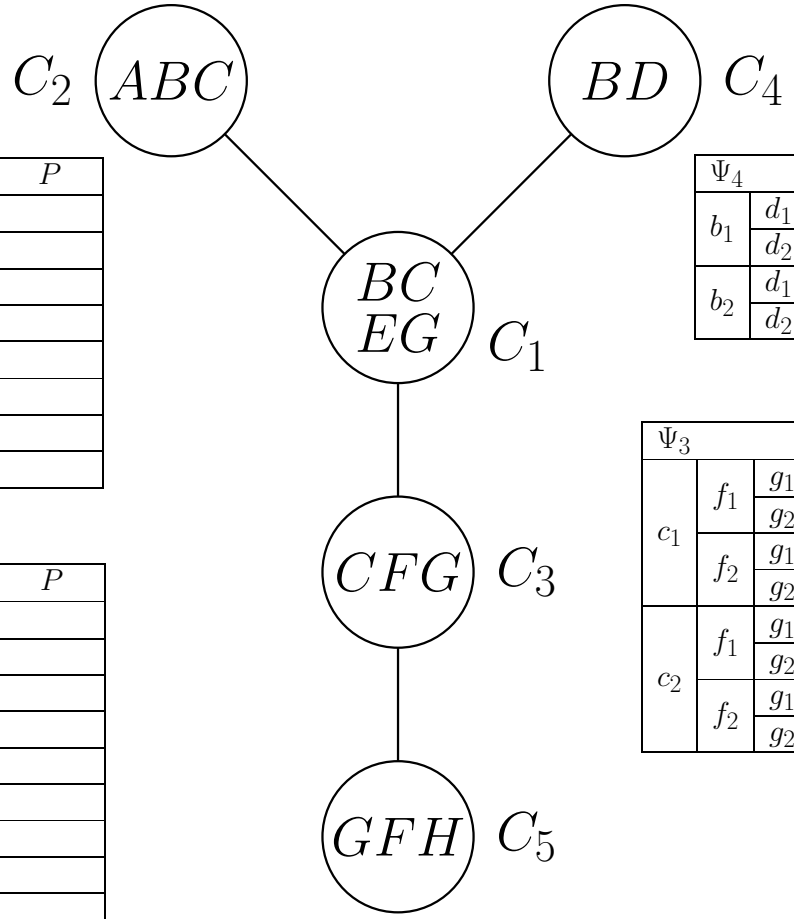
$$M_{14}(b) = \sum_{c,e,g} \Psi_1(b, c, e, g) M_{21}(b, c) M_{31}(c, g)$$

$$M_{35}(f, g) = \sum_{c} \Psi_3(c, f, g) M_{13}(c, g)$$



Arrows represent is-needed-for relations. Messages on the same level can be computed in any order. Messages are computed levelwise from top to bottom.

Example: Step 3: Initialization (Potential Layouts)



Ψ_2				P
a_1	b_1	c_1		
		c_2		
	b_2	c_1		
		c_2		
a_2	b_1	c_1		
		c_2		
	b_2	c_1		
		c_2		

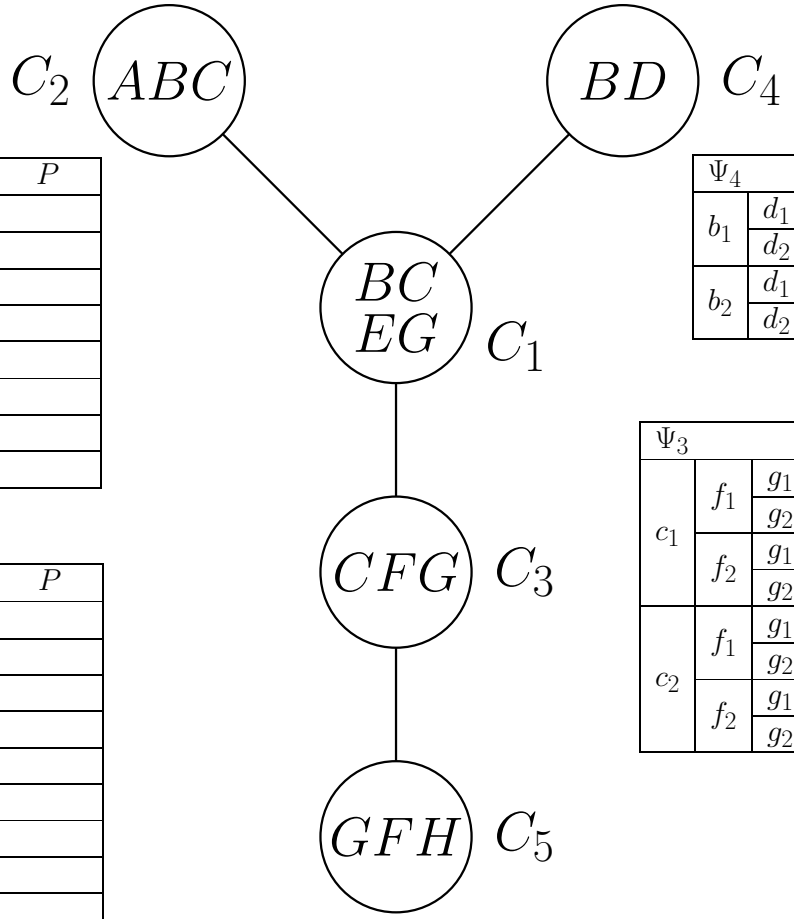
Ψ_4			P
b_1	d_1		
	d_2		
b_2	d_1		
	d_2		

Ψ_3				P
c_1	f_1	g_1		
		g_2		
	f_2	g_1		
		g_2		
c_2	f_1	g_1		
		g_2		
	f_2	g_1		
		g_2		

Ψ_1					P
b_1	c_1	e_1	g_1		
			g_2		
		e_2	g_1		
			g_2		
	c_2	e_1	g_1		
			g_2		
		e_2	g_1		
			g_2		
b_2	c_1	e_1	g_1		
			g_2		
		e_2	g_1		
			g_2		
	c_2	e_1	g_1		
			g_2		
		e_2	g_1		
			g_2		

Ψ_5				P
f_1	g_1	h_1		
		h_2		
	g_2	h_1		
		h_2		
f_2	g_1	h_1		
		h_2		
	g_2	h_1		
		h_2		

Example: Step 3: Initialization (Potential Values)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

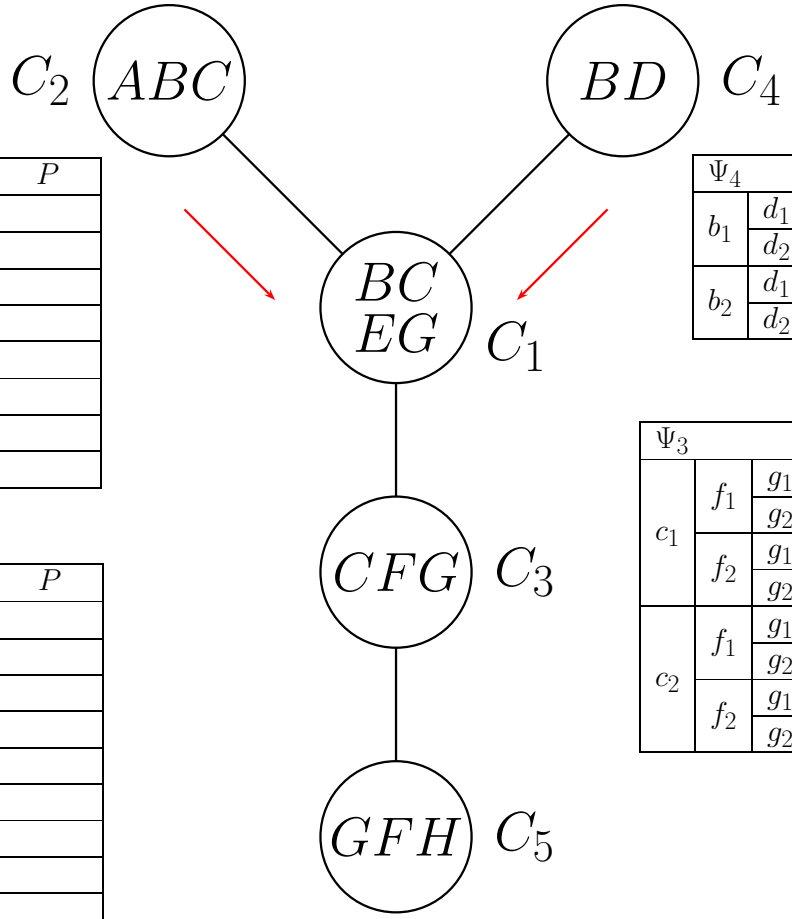
Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0.8	
	g_2	h_1	0.5	
		h_2	0.5	
f_2	g_1	h_1	0.4	
		h_2	0.6	
	g_2	h_1	0.7	
		h_2	0.3	

Example: Step 3: Initialization (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

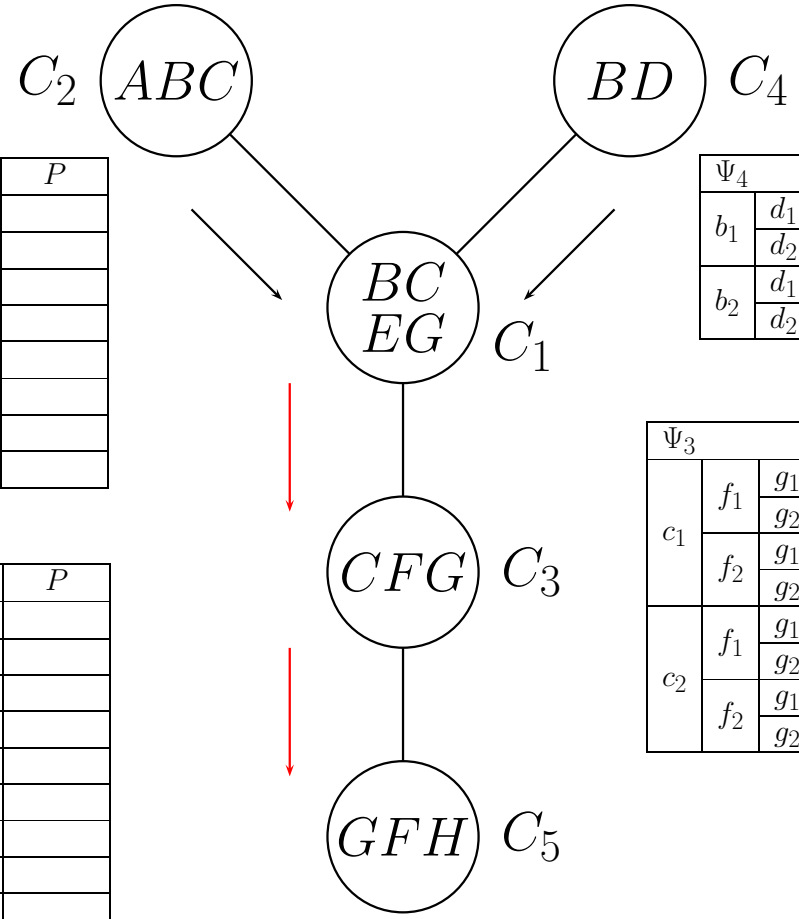
Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0.8	
	g_2	h_1	0.5	
		h_2	0.5	
f_2	g_1	h_1	0.4	
		h_2	0.6	
	g_2	h_1	0.7	
		h_2	0.3	

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

Example: Step 3: Initialization (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0.8	
	g_2	h_1	0.5	
		h_2	0.5	
f_2	g_1	h_1	0.4	
		h_2	0.6	
	g_2	h_1	0.7	
		h_2	0.3	

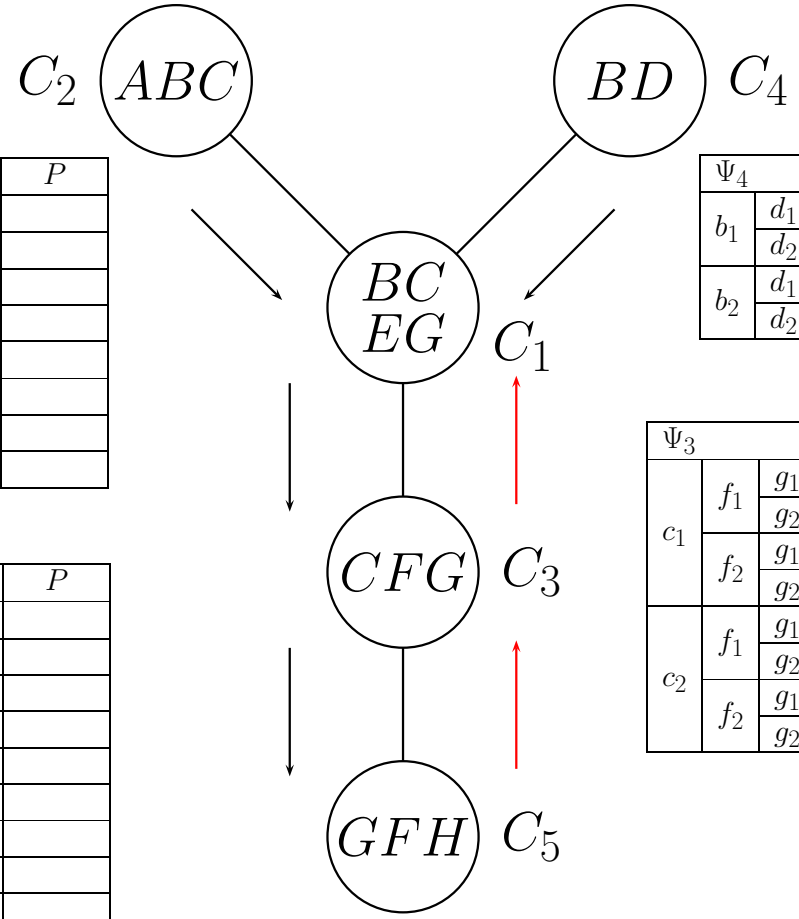
$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

Example: Step 3: Initialization (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0.8	
	g_2	h_1	0.5	
		h_2	0.5	
f_2	g_1	h_1	0.4	
		h_2	0.6	
	g_2	h_1	0.7	
		h_2	0.3	

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

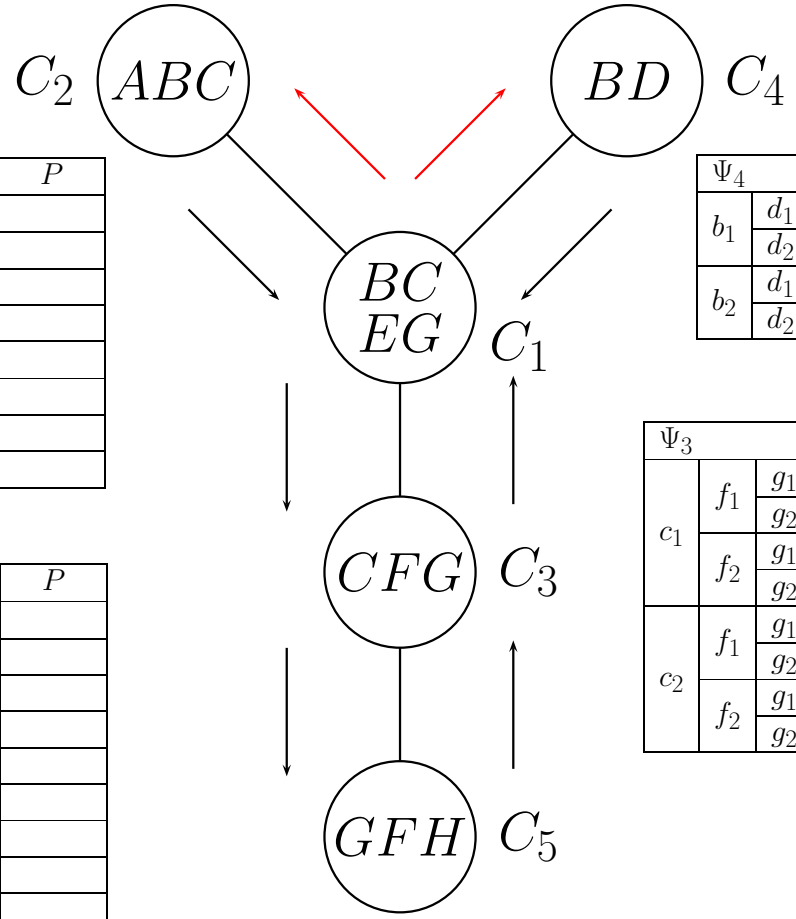
$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 1 & , 1 & , 1 & , 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 1 & , 1 & , 1 & , 1 \end{pmatrix}$$

Example: Step 3: Initialization (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0.8	
	g_2	h_1	0.5	
		h_2	0.5	
f_2	g_1	h_1	0.4	
		h_2	0.6	
	g_2	h_1	0.7	
		h_2	0.3	

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

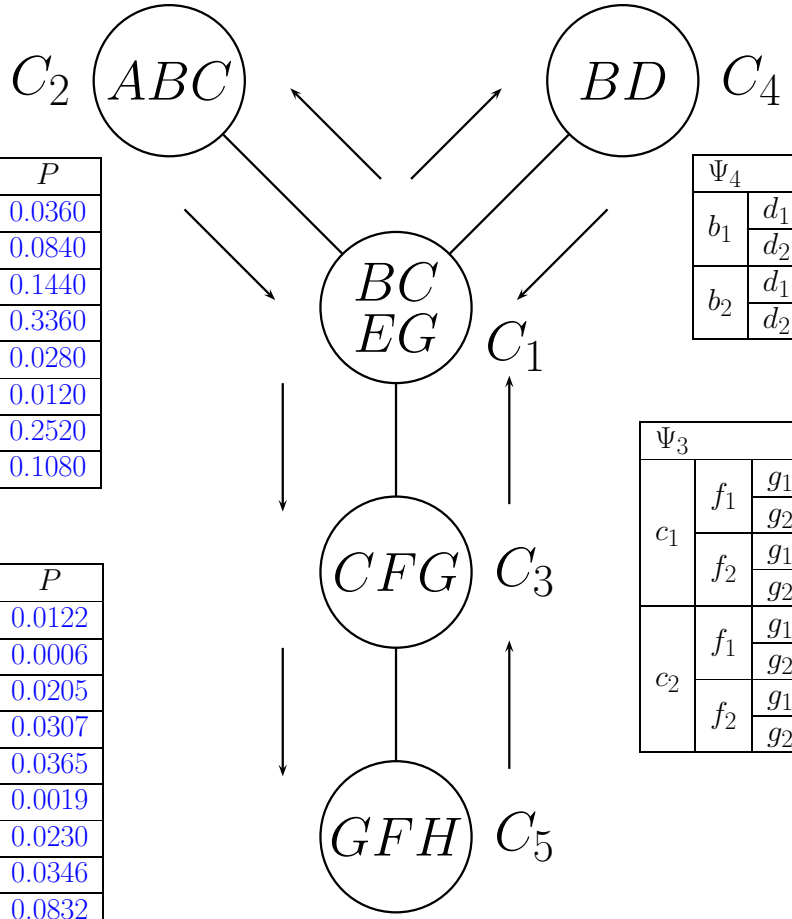
$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 1 & , 1 & , 1 & , 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 1 & , 1 & , 1 & , 1 \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 1 & , 1 & , 1 & , 1 \end{pmatrix}$$

$$M_{14} = \begin{pmatrix} b_1 & b_2 \\ 0.16, & 0.84 \end{pmatrix}$$

Example: Step 3: Initialization Complete



Ψ_2				P
a_1	b_1	c_1	0.036	0.0360
		c_2	0.084	0.0840
	b_2	c_1	0.144	0.1440
		c_2	0.336	0.3360
a_2	b_1	c_1	0.028	0.0280
		c_2	0.012	0.0120
	b_2	c_1	0.252	0.2520
		c_2	0.108	0.1080

Ψ_4			P
b_1	d_1	0.4	0.0640
	d_2	0.6	0.0960
b_2	d_1	0.7	0.5880
	d_2	0.3	0.2520

Ψ_3				P
c_1	f_1	g_1	0.1	0.0254
		g_2	0.1	0.0206
	f_2	g_1	0.9	0.2290
		g_2	0.9	0.1850
c_2	f_1	g_1	0.4	0.1162
		g_2	0.4	0.0998
	f_2	g_1	0.6	0.1742
		g_2	0.6	0.1498

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	0.0122
			g_2	0.010	0.0006
		e_2	g_1	0.320	0.0205
			g_2	0.480	0.0307
	c_2	e_1	g_1	0.380	0.0365
			g_2	0.020	0.0019
		e_2	g_1	0.240	0.0230
			g_2	0.360	0.0346
b_2	c_1	e_1	g_1	0.210	0.0832
			g_2	0.090	0.0356
		e_2	g_1	0.350	0.1386
			g_2	0.350	0.1386
	c_2	e_1	g_1	0.070	0.0311
			g_2	0.030	0.0133
		e_2	g_1	0.450	0.1998
			g_2	0.450	0.1998

Ψ_5				P
f_1	g_1	h_1	0.2	0.0283
		h_2	0.8	0.1133
	g_2	h_1	0.5	0.0602
		h_2	0.5	0.0602
f_2	g_1	h_1	0.4	0.1613
		h_2	0.6	0.2419
	g_2	h_1	0.7	0.2344
		h_2	0.3	0.1004

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1, & 1 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

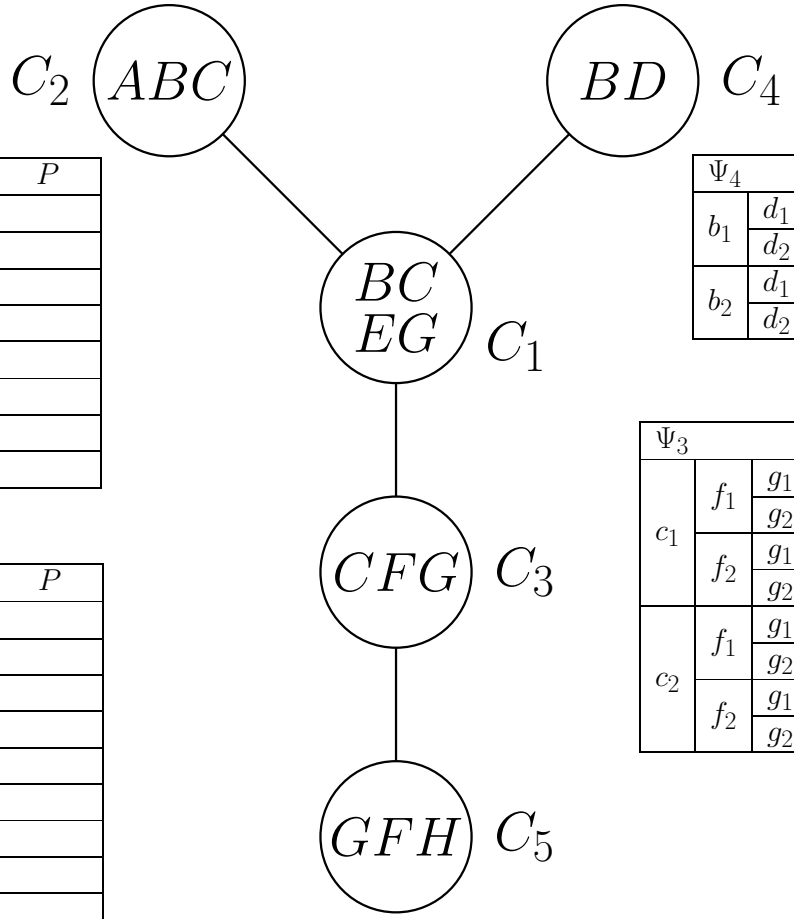
$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$M_{14} = \begin{pmatrix} b_1 & b_2 \\ 0.16, & 0.84 \end{pmatrix}$$

P	A	B	C	D	E	F	G	H
$\cdot_1$	**0.6000**	**0.1600**	**0.4600**	**0.6520**	**0.2144**	**0.2620**	**0.5448**	**0.4842**
$\cdot_2$	**0.4000**	**0.8400**	**0.4500**	**0.3480**	**0.7856**	**0.7380**	**0.4552**	**0.5158**

Example: Step 4: Evidence $H = h_1$ (Altering Potentials)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

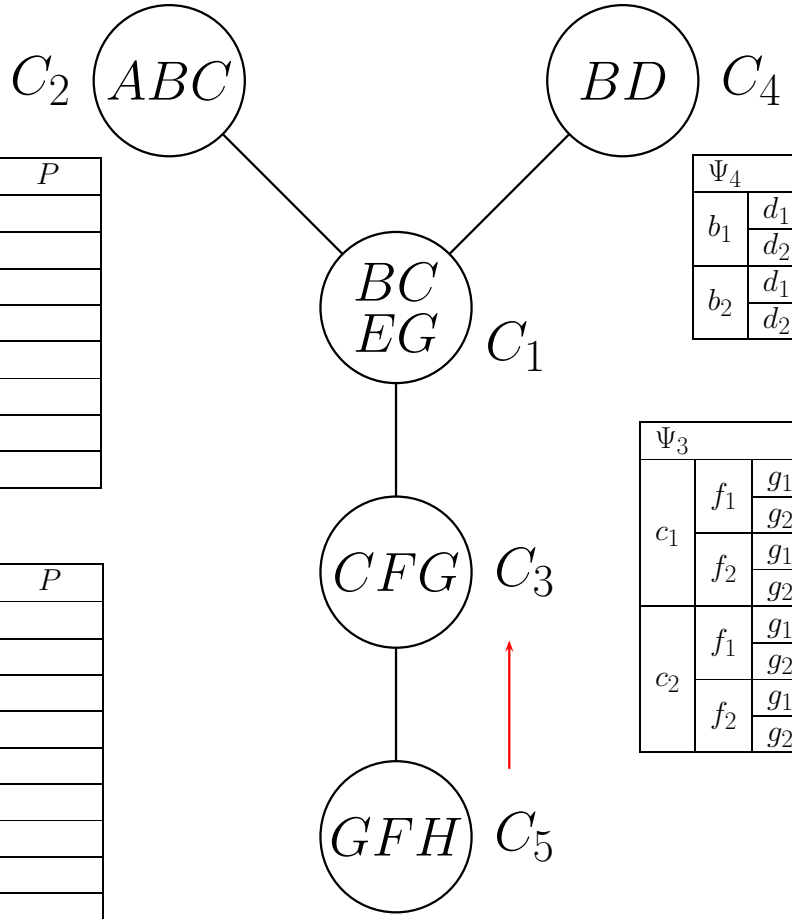
Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	**0**	
	g_2	h_1	0.5	
		h_2	**0**	
f_2	g_1	h_1	0.4	
		h_2	**0**	
	g_2	h_1	0.7	
		h_2	**0**	

Example: Step 4: Evidence $H = h_1$ (Sending Messages)



$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2 & 0.5 & 0.4 & 0.7 \end{pmatrix}$$

Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

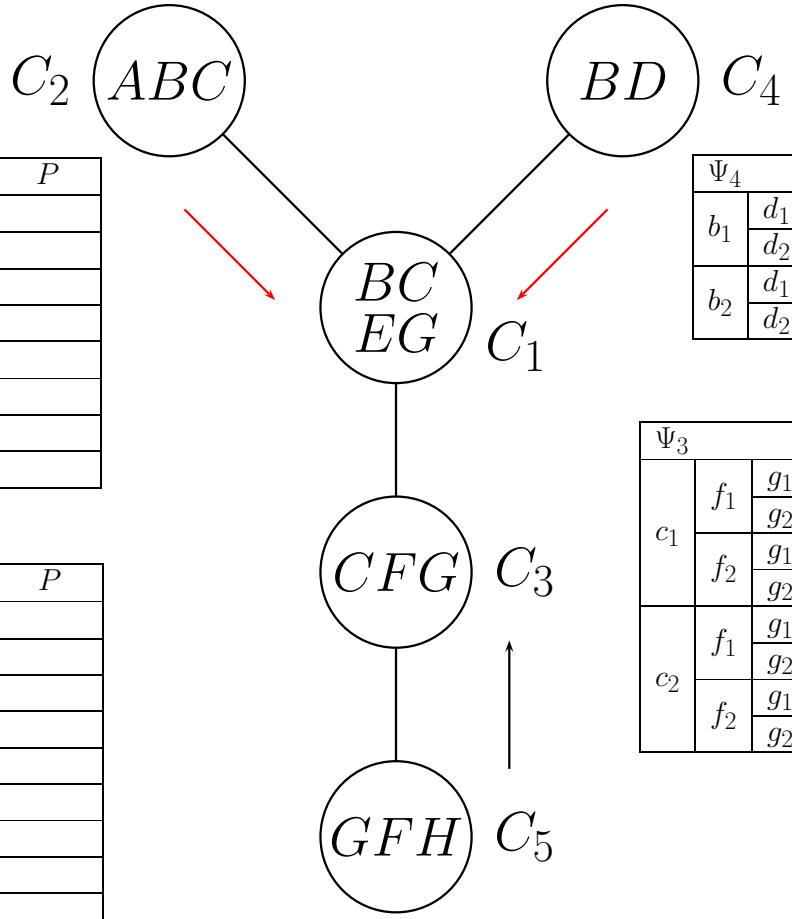
Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0	
	g_2	h_1	0.5	
		h_2	0	
f_2	g_1	h_1	0.4	
		h_2	0	
	g_2	h_1	0.7	
		h_2	0	

Example: Step 4: Evidence $H = h_1$ (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

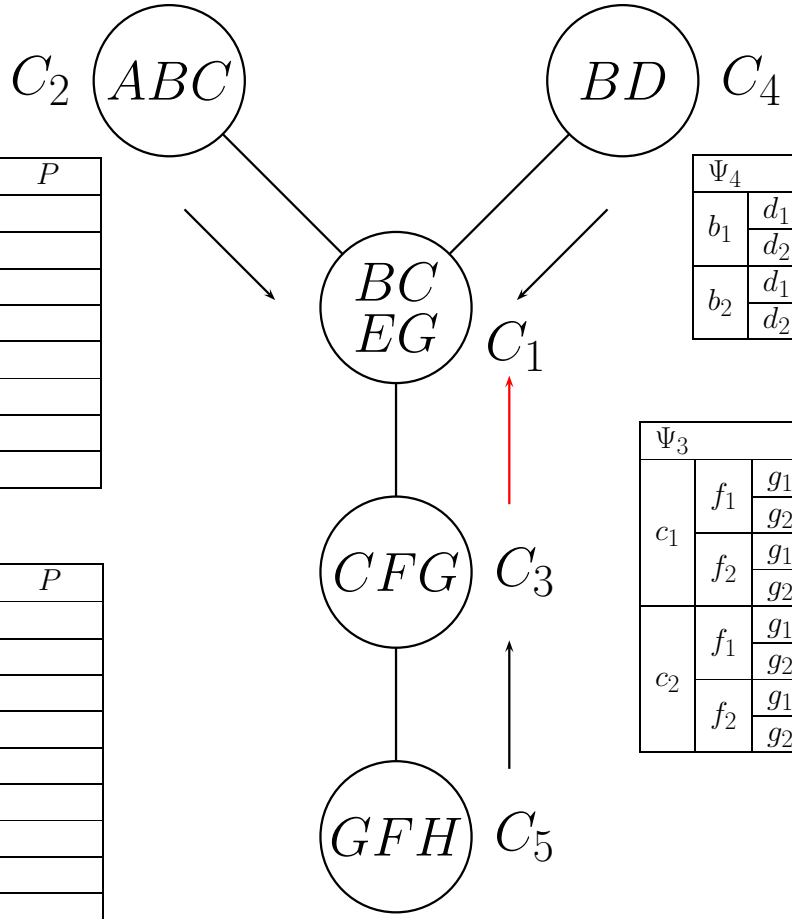
Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0	
	g_2	h_1	0.5	
		h_2	0	
f_2	g_1	h_1	0.4	
		h_2	0	
	g_2	h_1	0.7	
		h_2	0	

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2\,, & 0.5\,, & 0.4\,, & 0.7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1\,, & 1 \end{pmatrix}$$

Example: Step 4: Evidence $H = h_1$ (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0	
	g_2	h_1	0.5	
		h_2	0	
f_2	g_1	h_1	0.4	
		h_2	0	
	g_2	h_1	0.7	
		h_2	0	

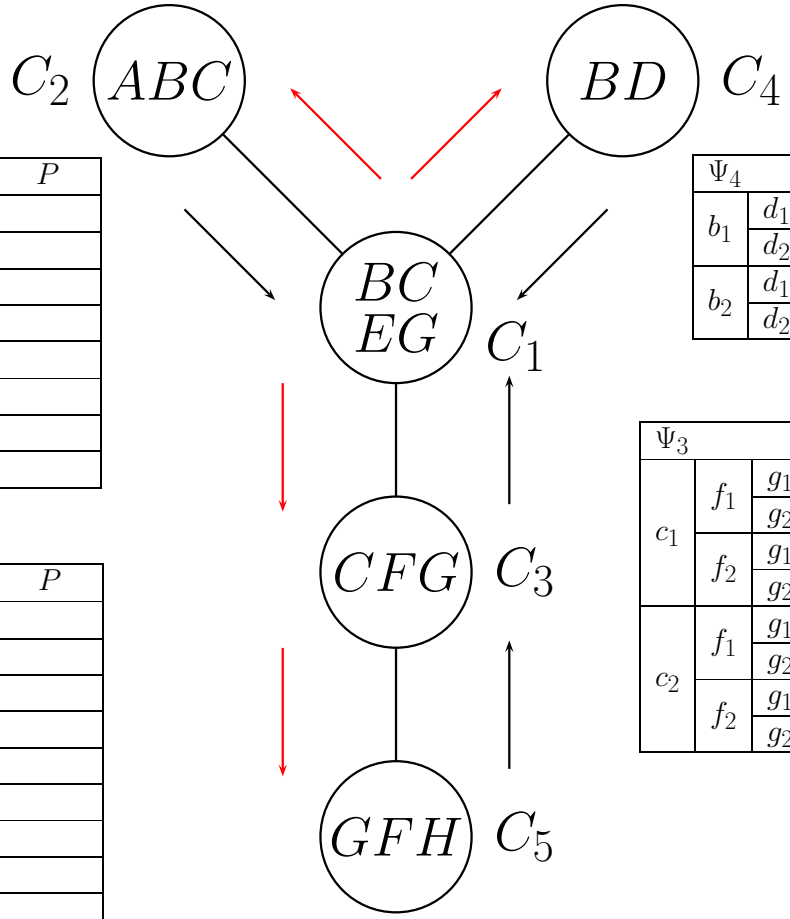
$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2\,, & 0.5\,, & 0.4\,, & 0.7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1\,, & 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.38, & 0.68, & 0.32, & 0.62 \end{pmatrix}$$

Example: Step 4: Evidence $H = h_1$ (Sending Messages)



Ψ_2				P
a_1	b_1	c_1	0.036	
		c_2	0.084	
	b_2	c_1	0.144	
		c_2	0.336	
a_2	b_1	c_1	0.028	
		c_2	0.012	
	b_2	c_1	0.252	
		c_2	0.108	

Ψ_4			P
b_1	d_1	0.4	
	d_2	0.6	
b_2	d_1	0.7	
	d_2	0.3	

Ψ_3				P
c_1	f_1	g_1	0.1	
		g_2	0.1	
	f_2	g_1	0.9	
		g_2	0.9	
c_2	f_1	g_1	0.4	
		g_2	0.4	
	f_2	g_1	0.6	
		g_2	0.6	

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	
			g_2	0.010	
		e_2	g_1	0.320	
			g_2	0.480	
	c_2	e_1	g_1	0.380	
			g_2	0.020	
		e_2	g_1	0.240	
			g_2	0.360	
b_2	c_1	e_1	g_1	0.210	
			g_2	0.090	
		e_2	g_1	0.350	
			g_2	0.350	
	c_2	e_1	g_1	0.070	
			g_2	0.030	
		e_2	g_1	0.450	
			g_2	0.450	

Ψ_5				P
f_1	g_1	h_1	0.2	
		h_2	0	
	g_2	h_1	0.5	
		h_2	0	
f_2	g_1	h_1	0.4	
		h_2	0	
	g_2	h_1	0.7	
		h_2	0	

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2\,, & 0.5\,, & 0.4\,, & 0.7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1\,, & 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.38, & 0.68, & 0.32, & 0.62 \end{pmatrix}$$

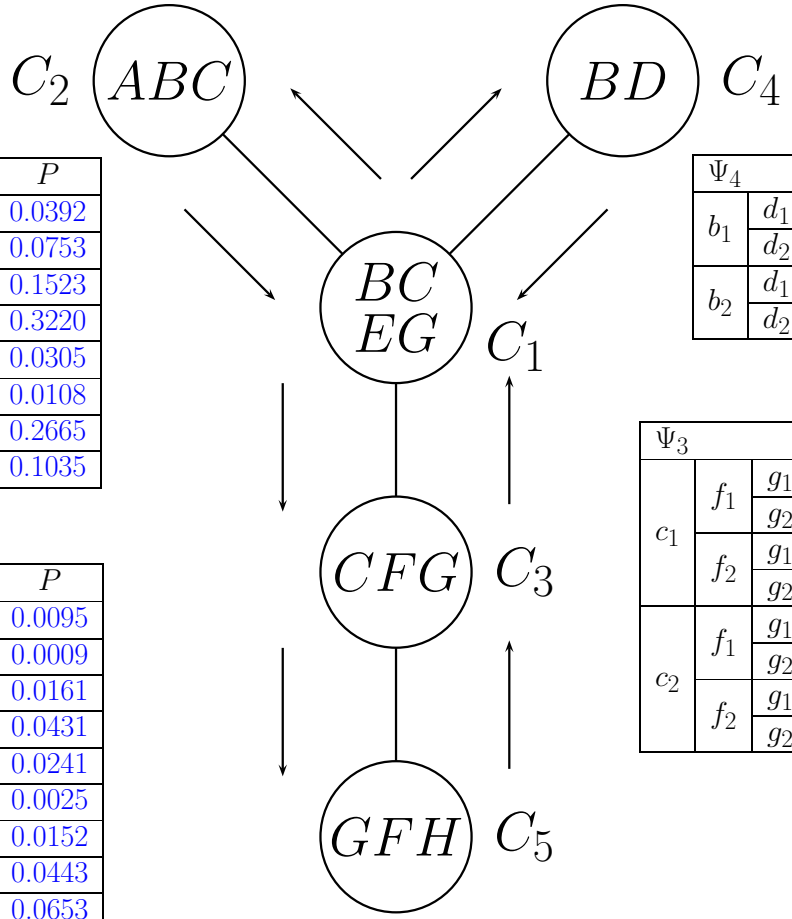
$$M_{12} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.527, & 0.434, & 0.512, & 0.464 \end{pmatrix}$$

$$M_{14} = \begin{pmatrix} b_1 & b_2 \\ 0.075, & 0.409 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

Example: Step 4: Evidence $H = h_1$ Incorporated



Ψ_2				P
a_1	b_1	c_1	0.036	0.0392
		c_2	0.084	0.0753
	b_2	c_1	0.144	0.1523
		c_2	0.336	0.3220
a_2	b_1	c_1	0.028	0.0305
		c_2	0.012	0.0108
	b_2	c_1	0.252	0.2665
		c_2	0.108	0.1035

Ψ_4			P
b_1	d_1	0.4	0.0623
	d_2	0.6	0.0934
b_2	d_1	0.7	0.5910
	d_2	0.3	0.2533

Ψ_3				P
c_1	f_1	g_1	0.1	0.0105
		g_2	0.1	0.0212
	f_2	g_1	0.9	0.1892
		g_2	0.9	0.2675
c_2	f_1	g_1	0.4	0.0480
		g_2	0.4	0.1031
	f_2	g_1	0.6	0.1440
		g_2	0.6	0.2165

Ψ_1					P
b_1	c_1	e_1	g_1	0.190	0.0095
			g_2	0.010	0.0009
		e_2	g_1	0.320	0.0161
			g_2	0.480	0.0431
	c_2	e_1	g_1	0.380	0.0241
			g_2	0.020	0.0025
		e_2	g_1	0.240	0.0152
			g_2	0.360	0.0443
b_2	c_1	e_1	g_1	0.210	0.0653
			g_2	0.090	0.0501
		e_2	g_1	0.350	0.1088
			g_2	0.350	0.1947
	c_2	e_1	g_1	0.070	0.0205
			g_2	0.030	0.0171
		e_2	g_1	0.450	0.1321
			g_2	0.450	0.2559

Ψ_5				P
f_1	g_1	h_1	0.2	0.0585
		h_2	0	0
	g_2	h_1	0.5	0.1243
		h_2	0	0
f_2	g_1	h_1	0.4	0.3331
		h_2	0	0
	g_2	h_1	0.7	0.4841
		h_2	0	0

$$M_{53} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.2 & 0.5 & 0.4 & 0.7 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.06, & 0.10, & 0.40, & 0.44 \end{pmatrix}$$

$$M_{41} = \begin{pmatrix} b_1 & b_2 \\ 1 & 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.38, & 0.68, & 0.32, & 0.62 \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} b_1,c_1 & b_1,c_2 & b_2,c_1 & b_2,c_2 \\ 0.527, & 0.434, & 0.512, & 0.464 \end{pmatrix}$$

$$M_{14} = \begin{pmatrix} b_1 & b_2 \\ 0.075, & 0.409 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} c_1,g_1 & c_1,g_2 & c_2,g_1 & c_2,g_2 \\ 0.254, & 0.206, & 0.290, & 0.250 \end{pmatrix}$$

$$M_{35} = \begin{pmatrix} f_1,g_1 & f_1,g_2 & f_2,g_1 & f_2,g_2 \\ 0.14, & 0.12, & 0.40, & 0.33 \end{pmatrix}$$

P	A	B	C	D	E	F	G	H
$\cdot_1$	**0.5888**	**0.1557**	**0.4884**	**0.6533**	**0.1899**	**0.1828**	**0.3916**	**1.0000**
$\cdot_2$	**0.4112**	**0.8443**	**0.5116**	**0.3467**	**0.8101**	**0.8172**	**0.6084**	**0.0000**

Summary

There are several exact inference methods such as variable elimination, clique tree propagation or recursive conditioning. These algorithms have complexity that is exponential with networks tree width. Exact inference is NP-hard.

In very large applications it is often useful to introduce topological structural constraints or restrictions on conditional probabilities, i.e. bounded variance algorithms.

There are also approximative inference methods.